Facets of the Linear Ordering Polytope: a unification for the fence family through weighted graphs ⋆

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Abstract

The binary choice polytope appeared in the investigation of the binary choice problem formulated by Guilbaud (1953) and Block and Marschak (1960). It is nowadays known to be the same as the linear ordering polytope from operations research (Grötschel, Jünger and Reinelt, 1985). The central problem is to find facet-defining linear inequalities for the polytope. Fence inequalities constitute a prominent class of such inequalities (Cohen and Falmagne 1978, 1990; Grötschel, Jünger and Reinelt, 1985). Two different generalizations exist for this class: the reinforced fence inequalities (Leung and Lee, 1994; Suck, 1992) and the stability-critical fence inequalities (Koppen, 1995). Together with the fence inequalities, these inequalities form the fence family. Building on previous work on the biorder polytope (Christophe, Doignon and Fiorini, 2004), we provide a new class of inequalities which unifies all inequalities from the fence family. The proof is based on a projection of polytopes. The new class of facet-defining inequalities is related to a specific class of weighted graphs, whose definition relies on a curious extension of the stability number. We investigate this class of weighted graphs which generalize the stability-critical graphs.

Key words: binary choice polytope, linear ordering polytope, facet-defining inequality, fence inequality, stability-critical graph

1 Introduction

A well-known problem of mathematical psychology and economics asks for a characterization of the binary choice probabilities that are generated by random linear orderings of the alternatives (Guilbaud, 1953; Block and Marschak, 1960). This problem was turned into the search for all facet-defining inequalities of a certain (convex) polytope (Megiddo, 1977), thus dubbed the 'binary choice polytope'. On the other hand, a polytope called the 'linear ordering polytope' appeared in operations research as a tool for building an optimal solution to the linear ordering problem (Grötschel, Jünger, and Reinelt, 1985a,b). It took some time before it was realized that the two polytopes are one and the same (see for instance Suck, 1992). In both disciplines, the central problem is that of listing as many facet-defining inequalities as possible—geometrically, one simply asks for facets. Because the problem of finding an optimal linear ordering is known to be NP-hard (Karp, 1972), there is little hope that a complete list of all facets will ever be established. Nevertheless, it is interesting to produce new facets because each of them at the same time gives a new necessary condition for binary probabilities to admit a random representation, and can also be put to good use in the optimization problem. In contrast, the multiple choice problem astonishingly admits an explicit solution established by Falmagne (1978, 1979) (for a maybe more clarifying proof, see Fiorini, 2004).

The first general scheme of facets of the binary choice vs. linear ordering polytope was discovered both in mathematical psychology and in operations research. Cohen and Falmagne (1978, 1990) and Grötschel et al. (1985b) indeed introduced each on their own a family of facets which surpass the most obvious facets (although at some time in the past, the latter were thought to be the only ones). These facets are called the 'fence inequalities'. Some years later, two distinct generalizations were proposed. First, the introduction of weights in the basic fence inequalities produced the 'reinforced fence inequalities' (Leung and Lee, 1994; we notice that Suck, 1992, found later the same result but published it earlier, again an illustration of parallel developments). The second generalization led through several successive steps (McLennan, 1990; Fishburn, 1990; Koppen, 1991) to 'stability-critical fence inequalities'. Here

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appears a marvelous connection between two distinct topics, due to Koppen (1995): the latter inequalities are essentially in a one-to-one correspondence with 'stability-critical graphs' (the simplest case, the fence inequality, corresponds to complete graphs).

Our contribution consists in a unification of the above two generalizations of fence inequalities via weighted graphs. To any weighted graph, we associate an inequality that is valid for the linear ordering polytope. These inequalities, which we call 'graphical inequalities', were first studied in the context of 'biorder polytopes' by Christophe, Doignon, and Fiorini (2004). When a graphical inequality defines a facet of the linear ordering polytope, the corresponding weighted graph is called a 'facet-defining graph', or FDG in short. Since FDGs generalize stability-critical graphs, we survey in Section 6 known results about the latter graphs. In particular, we emphasize the role of the 'defect' in attempts to classify stability-critical graphs. Section 7 is devoted to basic results on FDGs, some of them taken from Christophe, Doignon, and Fiorini (2004). The following section introduces the 'defect' of a FDG and establishes several of its properties. It concludes with the first steps into the classification of FDGs with a small defect.

To summarize, our contribution goes beyond providing a common generalization for the fence family. We also establish a list of properties of the corresponding FDGs. In this respect, our work is parallel to that of Lipták and Lovász (2000, 2001) who also investigate a generalization of stability-critical graphs in connection with (other) polytopes. Before focusing on FDGs, we formally describe in Sections 2 and 3 the fence family and the graphical inequalities, respectively. Then in Section 4 we collect prerequisites on biorders, relying on Doignon, Ducamp, and Falmagne (1984). Section 5 introduces a projection from the linear ordering polytope onto the biorder polytope. The projection is then used to prove that a graphical inequality is facet-defining for the linear ordering polytope if and only if it is facet-defining for the biorder polytope, except in one particular case.

This paper is heavily influenced by the work of Jean-Claude Falmagne. The senior author (J.-P. D.) was exposed by him to biorders in 1980 and to the binary choice polytope in 1988. All three authors are glad to dedicate their present contribution to Jean-Claude.

2 Background and the Fence Family

Let X, Y be finite sets, and let R denote a relation from X to Y. As relations are always considered in this paper as sets of ordered pairs, R is a subset of $X \times Y$. We often use ij as an abbreviation for (i, j) and write $i\,R\,j$ when the pair ij belongs to the relation R. In order to encode R geometrically, we resort to the real vector space $\mathbb{R}^{X\times Y}$, which has one coordinate per element of $X \times Y$. The coordinate of the pair ij is denoted by x_{ij}. The *characteristic vector* of R is the vector x^R in $\mathbb{R}^{X\times Y}$ such that $x^R_{ij} = 1$ if $ij \in R$ and $x^R_{ij} = 0$ otherwise.

Now let Z be a third finite set. A *linear ordering* on Z is a reflexive, transitive, antisymmetric and complete relation on Z, i.e., from Z to Z. The *binary choice polytope* or *linear ordering polytope* is defined as the convex hull in $\mathbb{R}^{Z\times Z} = \mathbb{R}^{Z^2}$ of the characteristic vectors x^L of all linear orderings L on Z. We denote it by P^Z_{LO}. Formally, we have

$$P^Z_{\mathrm{LO}} = \mathrm{conv}\{x^L \in \mathbb{R}^{Z^2} \mid L \text{ is a linear ordering on } Z\}. \tag{1}$$

The linear ordering polytope has precisely one vertex per linear ordering on Z. Note that the whole polytope lies inside the affine subspace defined by the equations $x_{ii} = 1$ for $i \in Z$ and $x_{ij} + x_{ji} = 1$ for $i, j \in Z$, $i \neq j$. Because these equations form a complete and irredundant system of equations for the polytope, we have $\dim P^Z_{\mathrm{LO}} = |Z|(|Z|-1)/2$. Consequently, there is no unique way to write a facet-defining inequality. We remark that besides the obvious symmetries derived from the permutations of the base set Z or from the reversal of all linear orderings, the polytope admits 'strange' symmetries found by McLennan (1990) and Bolotashvili, Kovalev, and Girlich (1999). The full group of symmetries was characterized by Fiorini (2001).

Classes of facet-defining inequalities for the linear ordering polytope are now described. Because they are all related to the fence inequalities (defined below), we collectively refer to them as the *fence family*. In the rest of the section, V and W are disjoint subsets of Z with the same cardinality, and f is a bijective mapping from V to W. A *fence inequality* is any inequality of the form

$$\sum_{i\in V} x_{if(i)} - \sum_{\substack{i\in V, j\in W\\ j\neq f(i)}} x_{ij} \leq 1. \tag{2}$$

Notice that, traditionally, the fence inequality is written in another, equivalent form. This inequality was independently discovered by Grötschel, Jünger, and Reinelt (1985a) and by Cohen and Falmagne (1990). Although the latter reference was published five years after the former, the working paper version dates back to 1978.

Proposition 1 (Grötschel et al., 1985a; Cohen and Falmagne, 1990) *The fence inequality (2) defines a facet of the linear ordering polytope P^Z_{LO} whenever $|Z| \geq 2|V| = 2|W| \geq 6$.*

A first idea to generalize the fence inequalities is to multiply all the terms of the form $x_{if(i)}$ in Inequality (2) by an integer t with $1 \leq t \leq |V| - 2$. The resulting inequality,

$$\sum_{i\in V} t\, x_{if(i)} - \sum_{\substack{i\in V, j\in W\\ j\neq f(i)}} x_{ij} \leq \frac{t(t+1)}{2}, \tag{3}$$

is called a *reinforced fence inequality*. Although these inequalities were given a name by Leung and Lee (1994), they were implicitly known before as special cases of Gilboa's 'diagonal inequalities' (Gilboa, 1990, working paper of 1985). They were independently discovered also by Suck (1992).

Proposition 2 (Leung and Lee, 1994; Suck, 1992) *The reinforced fence inequality (3) defines a facet of the linear ordering polytope P_{LO}^{Z} whenever $|Z| \geq 2|V| = 2|W| \geq 6$ and $1 \leq t \leq |V| - 2$.*

A second generalization of the fence inequality, due to Koppen (1995), arises when the complete graph implicit in the structure of the fence inequality is replaced by an arbitrary graph. Let thus G be any graph with vertex set $V(G)$ and edge set $E(G)$ (for graph terminology, we usually follow Diestel, 2005). The *stability number* $\alpha(G)$ of G is the largest cardinality of a stable subset of $V(G)$, where a set of vertices is *stable* if its vertices are mutually nonadjacent. Considering as before V and W, two disjoint subsets of Z with the same cardinality, and a bijective mapping $f : V \to W$, we assume now $V = V(G)$. The inequality

$$\sum_{v\in V(G)} x_{vf(v)} - \sum_{\{v,w\}\in E(G)} (x_{vf(w)} + x_{wf(v)}) \leq \alpha(G) \tag{4}$$

is easily seen to be valid for the linear ordering polytope. Koppen (1995) gave the following characterization of the graphs G for which Inequality (4) is facet-defining.

Proposition 3 (Koppen, 1995) *Inequality (4) defines a facet of the linear ordering polytope if and only if G is the one-vertex graph or G has at least three vertices, is connected and stability-critical.*

A graph G without isolated vertex is said to be *stability-critical* if its stability number increases whenever an edge is removed from its edge set. When G satisfies the conditions of Proposition 3, we call Inequality (4) a *stability-critical fence inequality*.

Observe that when G is a complete graph with at least three vertices, Inequality (4) is a fence inequality. Hence stability-critical fence inequalities generalize fence inequalities. Two more special cases of stability-critical fence inequalities have been considered in the literature. The first special case occurs when G is an odd cycle. The corresponding inequalities were discovered independently by Grötschel et al. (1985a), McLennan (1990) and Fishburn (1990). The second special case, which subsumes the first, occurs when G is the graph C_n^ℓ with vertex set $V(C_n^\ell) = \{1, 2, \ldots, n\}$ and edge set

$$E(C_n^\ell) = \{\{i, j\} \mid i, j \in V, 0 < \min\{|i-j|, |k-i-j|\} \leq \ell\}, \tag{5}$$

and with $3 \leq 2\ell + 1 \leq n$. The corresponding inequalities were investigated independently by Bolotashvili (1987) under the name $(n, \ell+1)$-*fence inequalities*, and by Koppen (1991). It is known that $\alpha(C_n^\ell) = \lfloor n/(\ell+1) \rfloor$ and that C_n^ℓ is stability-critical if and only if $\ell + 1$ divides $n + 1$. Apparently, Bolotashvili (1987) showed that the $(n, \ell + 1)$-fence inequalities define facets of the linear ordering polytope, provided that $\ell + 1$ divides $n + 1$.

We remark that by applying certain nonobvious symmetries of the linear ordering polytope to the facet-defining inequalities given above, one obtains new facet-defining inequalities. Some of them were studied in the literature, as for instance the augmented fence inequalities of McLennan (1990) and Leung and Lee (1994), and the augmented reinforced fence inequalities of Leung and Lee (1994). Other examples are given in Bolotashvili, Kovalev, and Girlich (1999).

3 Graphical Inequalities

In the preceding section, we have seen two different generalizations of the fence inequalities. The first changes the coefficients of the positive terms and the second changes the structure of the inequality by substituting any graph for the complete graph. It is quite natural to combine both generalizations, which is precisely what is done in this section.

A *weighted graph* is a pair (G, μ) where G is a graph and μ is a function assigning an integral weight $\mu(v)$ to each vertex v of G. Let S be any subset of the vertex set of G. We denote by $\mu(S) = \sum_{v \in S} \mu(v)$ the total weight of S. The *worth* (or *net weight*) of S is the difference between the total weight of S and the number of edges in the subgraph of G induced by S. This number of edges, denoted as $||G[S]||$ in Diestel (2005), will be given here by the simpler notation $||S||$. Thus the worth of S equals

$$w(S) = \mu(S) - ||S||. \tag{6}$$

If S is of maximum worth amongst subsets of $V(G)$ we say that S is *tight*. We define $\alpha(G, \mu)$ to be the worth of a tight set in (G, μ). That is, we let

$$\alpha(G, \mu) = \max_{S \subseteq V(G)} w(S). \tag{7}$$

When $\mu = \mathbb{1}$, i.e., when the weight of each vertex is 1, we have $\alpha(G, \mathbb{1}) = \alpha(G)$. Hence the parameter $\alpha(G, \mu)$ can be considered as a generalization of the stability number of a graph to weighted graphs.

Let (G,μ) be a weighted graph whose vertex set $V = V(G)$ is contained in Z. We again assume that f is a bijection from V to a subset of Z which is disjoint from V. The *graphical inequality* of (G,μ) reads

$$\sum_{v \in V(G)} \mu(v) x_{vf(v)} \quad - \sum_{\{v,w\} \in E(G)} (x_{vf(w)} + x_{wf(v)}) \quad \leq \quad \alpha(G,\mu). \tag{8}$$

Because of the choice of the right-hand side, the inequality is always valid for the linear ordering polytope P^Z_{LO}. When $\mu = \mathbb{1}$, it is identical to Inequality (4). Moreover, when G is a complete graph and $\mu = t\mathbb{1}$ with $1 \leq t \leq |V| - 2$, the graphical inequality is a reinforced fence inequality.

We say that a weighted graph (G,μ) is *facet-defining* when its graphical inequality defines a facet of the linear ordering polytope, and moreover G has at least three vertices (thus discarding the single weighted graph $(K_1, \mathbb{1})$ makes later statements simpler). A vertex of a weighted graph is said to be *degenerated* if both its weight and its degree equal zero. In order to avoid trivial cases, we always assume that a weighted graph has no degenerated vertex. We think that understanding the structure of facet-defining graphs, in short FDGs, is a nice and important research problem. By Proposition 3, this class contains all connected stability-critical graphs except the complete graph K_2. We survey important results on stability-critical graphs in Section 6, and adapt some of these results to the more general case of facet-defining graphs in Sections 7 and 8. We also provide results about FDGs which are of a new type.

Before starting to investigate FDGs, we need to establish when a graphical inequality defines a facet of the linear ordering polytope. To this aim, we make use of another polytope, namely the 'biorder polytope'. In the next two sections, we remind the reader about biorders and the definition and some properties of the biorder polytope.

4 Biorders and the Biorder Polytope

Let X and Y be two finite sets. A relation B from X to Y is a *biorder* when

$$i\,B\,j \text{ and } k\,B\,\ell \text{ imply } i\,B\,\ell \text{ or } k\,B\,j \tag{9}$$

for all elements $i, k \in X$ and $j, \ell \in Y$. While biorders received various names, for instance 'Guttman scales' (after Guttman, 1944), 'Ferrers relations' (Riguet, 1951), 'bi-quasi-series' (Ducamp and Falmagne, 1969), the term comes from Doignon, Ducamp, and Falmagne (1984) (where the case of infinite sets X and Y is also considered). The number of biorders from X to Y is a function of only $|X|$ and $|Y|$, which is investigated in Christophe, Doignon, and Fiorini (2003).

The *biorder polytope* $P_{\text{Bio}}^{X\times Y}$ was introduced in Christophe, Doignon, and Fiorini (2004), with a definition similar to that of the linear ordering polytope. Each biorder B from X to Y is encoded by its characteristic vector x^B, considered as an element of the space $\mathbb{R}^{X\times Y}$ (points in this space have one coordinate x_{ij} for each ordered pair ij in $X \times Y$). The convex hull of all points x^B in $\mathbb{R}^{X\times Y}$, for B any biorder from X to Y, is the *biorder polytope* $P_{\text{Bio}}^{X\times Y}$. The biorder polytope $P_{\text{Bio}}^{X\times Y}$ has dimension $|X| \cdot |Y|$.

The graphical inequality (cf. Equation (8)) has an even more natural definition for the biorder polytope $P_{\text{Bio}}^{X\times Y}$ than for the linear ordering polytope. Assume the weighted graph (G, μ) satisfies $V(G) \subseteq X$, and $f : V(G) \to Y$ is an injective mapping. The *graphical inequality* of (G, μ) for $P_{\text{Bio}}^{X\times Y}$ reads

$$\sum_{v\in V(G)} \mu(v)x_{vf(v)} \;-\; \sum_{\{v,w\}\in E(G)} (x_{vf(w)} + x_{wf(v)}) \quad\leq\quad \alpha(G,\mu). \tag{10}$$

The following results are adapted from Christophe et al. (2004). The graphical inequality is valid for $P_{\text{Bio}}^{X\times Y}$. It defines a facet if and only if the tight sets of (G, μ) satisfy a technical condition that we will formulate in Proposition 4 after having introduced some concepts. Let (G, μ) be a weighted graph. We denote by $E(S)$ the collection of edges contained in a set S of vertices. To each tight set T of (G, μ), we associate the affine equation

$$\sum_{v\in T} y_v + \sum_{e\in E(T)} y_e = \alpha(G,\mu). \tag{11}$$

We thus form the *system* of (G, μ), also described as $\mathcal{T} \cdot \mathcal{Y} = \mathcal{A}$, where the rows of the matrix $\mathcal{T}$ correspond to tight sets of (G, μ), the vector $\mathcal{Y}$ contains the real unknowns y_v and y_e for $v \in V(G)$ and $e \in E(G)$, and $\mathcal{A} = [\alpha(G,\mu)\ \alpha(G,\mu)\ \ldots\ \alpha(G,\mu)]^t$.

Proposition 4 (Christophe et al., 2004) *Let (G, μ) be a weighted graph with at least three vertices. The graphical inequality of (G, μ), as in Equation* (10)*, is facet-defining for the biorder polytope $P_{\text{Bio}}^{X\times Y}$ if and only if the system of (G, μ) has a unique solution.*

The vector y defined by $y_v = \mu(v)$ and $y_e = -1$ for all $v \in V(G)$, $e \in E(G)$ is always a solution to the system of (G, μ), so we require in Proposition 4 that there is no other solution.

Assuming some relationships among X, Y and Z, we now proceed to show that the graphical inequality is facet-defining for $P_{\text{Bio}}^{X\times Y}$ if and only if it is facet-defining for P_{LO}^{Z} (with one exception). A projection from P_{LO}^{Z} onto $P_{\text{Bio}}^{X\times Y}$ will be our main tool.

5 Projection of the Linear Ordering Polytope onto the Biorder Polytope

Let Z be any finite set, and X, Y be disjoint, nonempty subsets of Z. To any relation R on Z we associate the *induced relation* from X to Y, which is the intersection $R \cap (X \times Y)$. As we will now show, linear orderings on Z are then exactly mapped onto the biorders from X to Y.

Proposition 5 *Let X, Y and Z be as above. Any linear ordering on Z induces a biorder from X to Y. Conversely, every biorder from X to Y is induced by a linear ordering on Z.*

PROOF. The intersection of any linear ordering on Z with $X \times Y$ is a biorder from X to Y, as easily seen. Hence the first part of the proposition holds. To show the second part, let B be a biorder from X to Y. Then the relation R on Z obtained from B by adding all pairs $ji \in Y \times X$ with $ij \notin B$ is acyclic. Hence R is contained in some linear ordering L on Z. By the choice of R, the biorder from X to Y induced by L is exactly B. $\square$

We now build a projection from the linear ordering polytope P_{LO}^{Z} onto the biorder polytope $P_{\mathrm{Bio}}^{X \times Y}$. First define the linear projection

$$\pi : \mathbb{R}^{Z^2} \to \mathbb{R}^{X \times Y} : x \mapsto x' = \pi(x), \tag{12}$$

where $x'_{ij} = x_{ij}$ for $ij \in X \times Y$. From Proposition 5, we see at once that π maps the vertex set of the linear ordering polytope onto the vertex set of the biorder polytope. Indeed, π maps a vertex x^L of the linear ordering polytope to the vertex x^B of the biorder polytope, where $B = L \cap (X \times Y)$ is the biorder induced by L. As a consequence, π maps the linear ordering polytope P_{LO}^{Z} onto the biorder polytope $P_{\mathrm{Bio}}^{X \times Y}$. By the proof of Proposition 5, the vertices of the linear ordering polytope which are mapped by π to a given vertex x^B of the biorder polytope correspond to the linear extensions of an acyclic orientation of the complete bipartite graph with color classes X and Y determined by B.

We now switch to a more general setting in order to state and prove a lemma which is instrumental for showing the main result of this section. Let P and Q be two polytopes and let $\dot\rho : P \to Q$ denote a *projection of polytopes*, that is, the restriction to P of an affine map ρ from the space in which P is defined to the space in which Q is defined, mapping P onto Q. The projection $\dot\rho$ yields a lifting of the faces of Q to the faces of P: for every face F of Q the *preimage* $\dot\rho^{-1}(F) = \{x \in P \mid \rho(x) \in F\}$ is a face of P. Consider a face F of Q. The *plank* of F is the vector subspace defined by

$$\operatorname{plank} F \;=\; \operatorname{vect}\{q-p \mid p,q \in P \text{ and } \rho(p)=\rho(q)\in F\}, \tag{13}$$

where $\operatorname{vect} A$ denotes the vector subspace spanned by A. Note that the plank itself depends on a choice of origin in the ambient space of P, but its dimension is always the same. As we now show, this vector subspace is useful in computing the dimension of the preimage of a face.

Lemma 6 *For any face F of Q, we have*

$$\dim \dot{\rho}^{-1}(F) = \dim F + \dim \operatorname{plank} F. \tag{14}$$

PROOF. Let W and U denote the two affine subspaces spanned by F and its preimage, respectively. Let o be a point in the relative interior of $\dot{\rho}^{-1}(F)$. Taking o as an origin in U and its image $\rho(o)$ as an origin in W, we view U and W as vector spaces. The affine map ρ restricts to a linear mapping R from U onto W. As is easily verified, the plank of F computed with o as origin is simply the kernel of R. The lemma then follows from the well-known equation $\dim U = \dim \ker R + \dim W$. $\square$

Before turning to the main result of this section, we note the following lemma.

Lemma 7 *If the preimage $\dot{\rho}^{-1}(F)$ of a face F of Q is a facet of P, then F is itself a facet of Q.*

PROOF. If F is not a facet of Q then there exists a facet F' of Q which properly contains F. Now the preimage of F' properly contains the preimage of F, so the preimage of F' equals P, contradicting the fact that $\dot{\rho}$ is surjective. $\square$

We now go back to our initial case, where $P = P^{Z}_{\mathrm{LO}}$, $Q = P^{X\times Y}_{\mathrm{Bio}}$ and $\rho = \pi$. Let $\dot{\pi}$ denote the restriction of π to P^{Z}_{LO}. Thus $\dot{\pi}$ plays the role of $\dot{\rho}$. Moreover, we suppose $V \subseteq X$ with some injective mapping $f : V \to Y$.

Proposition 8 *Let (G,μ) be a weighted graph with $V = V(G)$. Assume the graphical inequality* (10) *of (G,μ) defines a facet F of $P^{X\times Y}_{\mathrm{Bio}}$. Then the preimage of F under $\dot{\pi}$ is a facet of P^{Z}_{LO}, unless $(G,\mu) = (K_2, \mathbb{1})$.*

The difference of the sets A and B will be denoted as $A \setminus B$.

PROOF. Since the assertion is easily verified when G has at most two vertices, we can assume that G has at least three vertices. In virtue of the trivial lifting lemma for biorder polytopes (Christophe et al., 2004), we may assume $V = X$ and $f(X) = Y$. Similarly, because of the trivial lifting lemma for linear ordering polytopes (Grötschel et al., 1985a), we may assume $Z = X \cup Y$. Setting $q = |Z|$, $m = |X|$, we now have $q = 2\,m$. It suffices then to prove the inequality

$$\dim \operatorname{plank} F \geq \dim P_{\mathrm{LO}}^{Z} - \dim P_{\mathrm{Bio}}^{X\times Y} = \frac{q(q-1)}{2} - m^2. \tag{15}$$

Indeed, this inequality together with Lemma 6 implies that the dimension of the preimage of F is at least that of a facet of the linear ordering polytope. On the other hand, the preimage of F is not the whole linear ordering polytope because F is a proper face and π is surjective, hence $\dot{\pi}^{-1}(F)$ is a facet of P_{LO}^{Z}.

Notice that the right-hand side of Equation (15) is the number of unordered pairs $\{k, k'\}$ such that $k, k' \in X$ or $k, k' \in Y$. We denote by $e_{kk'}$ the vector in the canonical basis of $\mathbb{R}^{Z^2}$ that corresponds to the pair kk'. Let us show that for all $i, i' \in X$ with $i \neq i'$, we get $e_{ii'} - e_{i'i} \in \operatorname{plank} F$. As a similar argument can be given for all $j, j' \in Y$ with $j \neq j'$, we are done because altogether these $2 \cdot m(m-1)$ vectors generate a vector subspace of dimension $m(m-1) = q(q-1)/2 - m^2$.

Case 1: $\{i, i'\} \in E(G)$. By Proposition 11(C3) in Christophe et al. (2004), or by Proposition 11(C4) of Section 7, there exists a tight set S avoiding both i and i'. Pick any linear ordering M on S and list the elements of S by increasing ranks as $s_1, s_2, \ldots, s_\ell$. Still abbreviating a pair as $(x, f(y))$ into $xf(y)$, we let $B = \{xf(y) \mid xy \in M\}$. Then B is a biorder from X to Y whose characteristic vector is a vertex of F, according to Proposition 6 in Christophe et al. (2004). Any linear ordering L on Z which has $Y \setminus f(S)$ as initial set, $X \setminus S$ as final set and which satisfies $s_1 \, L \, f(s_1) \, L \, s_2 \, L \, f(s_2) \, L \, \ldots \, L \, s_\ell \, L \, f(s_\ell)$ induces B on $X \times Y$. There exist two such linear orderings L_1 and L_2 with $L_1 \setminus L_2 = \{ii'\}$ and $L_2 \setminus L_1 = \{i'i\}$. It follows $x^{L_1} - x^{L_2} = e_{ii'} - e_{i'i} \in \operatorname{plank} F$.

Case 2: $\{i, i'\} \notin E(G)$. There exists some tight set S containing exactly one vertex in $\{i, i'\}$. This is true because if no such S existed, the system in Equation (13) of Christophe et al. (2004) would not have a unique solution, contradicting the fact that F is a facet (or see Proposition 11(C7) in Section 7). Without loss of generality, we assume $i \in S$ and $i' \notin S$. Let M be any linear ordering on S with i as its maximum, then $B = \{xf(y) \mid xy \in M\} \cup \{i'f(i)\}$ is a biorder from X to Y such that x^B is a vertex of F. The argument then goes as in the first case above. $\square$

Using Lemma 7, we can easily show that the converse of Proposition 8 also holds. Summarizing, we see that the following corollary holds.

Corollary 9 *A graphical inequality is facet-defining for the linear ordering polytope* P_{LO}^{Z} *if and only if it is facet-defining for the biorder polytope* $P_{\mathrm{Bio}}^{X\times Y}$*, except if the underlying weighted graph is* $(K_2, \mathbb{1})$.

6 Stability-Critical Graphs

In this section, we briefly survey important results concerning stability-critical graphs, thus complementing the report of Koppen (1995) (Section 7). We refer the reader to Lovász and Plummer (1986) (pages 445–456) and Lovász (1993) (pages 64–65) for a more detailed account.

We remind the reader that a graph is *stability-critical* if it has no isolated vertices and removing any of its edges increases its stability number. One of the first results concerning stability-critical graphs is due to Erdős and Gallai (1961). They introduced the *defect* $\delta(G) = |V(G)| - 2\,\alpha(G)$ of a graph G and proved its nonnegativity when G is stability-critical. The defect δ is a key parameter in the theory of stability-critical graphs. Hajnal (1965) established an upper bound of $\delta + 1$ on the degree of a vertex and Surányi (1975b) proved that equality in the previous bound can occur for at most $\delta + 2$ vertices if $\delta > 1$. Sewell and Trotter (1993) also proved that every connected, stability-critical graph with defect at least two contains an *odd subdivision* of K_4, that is, the graph K_4 where each edge is replaced by a path with an odd number of edges.

From a general point of view, stability-critical graphs exhibit many different structures and a satisfying characterization seems out of reach. Nevertheless, more insight was obtained by considering them for a fixed defect. Indeed, let G be a connected stability-critical graph (note that the assumption of connectedness is not really restrictive, since a non connected stability-critical graph consists of connected stability-critical components). If $\delta(G) = 0$, the theorem of Hajnal, recalled in previous paragraph, implies $G = K_2$. For $\delta(G) = 1$, it also implies that G is either a path or a cycle. Because paths and even cycles have defect less than 1, G must be an odd cycle or, equivalently, an odd subdivision of K_3. Andrásfai (1967) proved that $\delta(G) = 2$ occurs exactly when G is an odd subdivision of K_4. More generally, for each fixed natural number δ there is a finite set of graphs such that if the defect of G is δ then G is an odd subdivision of one of those. This was first proved for $\delta = 3$ by Surányi (1975b) and later for all δ by Lovász (1978).

As seen in the previous paragraph, odd subdivisions are useful for characterizing stability-critical graphs of a given defect. In its simplest form, an odd subdivision only *trisects* one edge, that is, it replaces the edge by a path composed of three edges. Thus, an odd subdivision can be seen as a composition

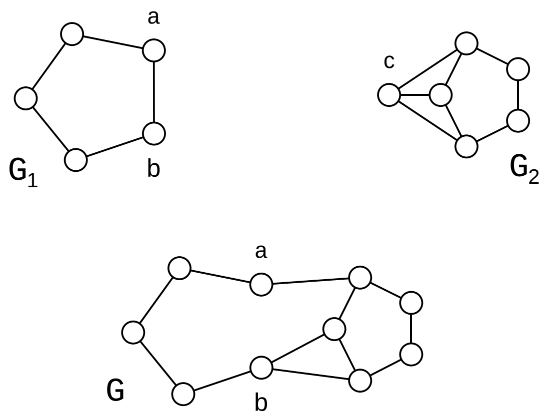

Fig. 1. An example of the construction of new stability-critical graphs.

of a certain number of trisections. It turns out that trisecting an edge is only but a special case of a more general method to construct a connected stability-critical graph by 'gluing' two smaller ones. In order to describe it, we need the fact that connected stability-critical graphs are also 2-connected (Lovász, 1993). Let G_1 and G_2 be two connected stability-critical graphs other than K_2 and choose an edge $\{a, b\}$ of G_1 and a vertex c of G_2. Define the graph G from of G_1 and G_2 as follows (an example is given in Figure 1):

- take the disjoint union of G_1 and G_2,
- remove the edge $\{a, b\}$,
- make each neighbor of c adjacent to exactly one vertex of $\{a, b\}$, ensuring that a becomes adjacent to at least one such neighbor, and the same for b,
- remove the vertex c and all edges containing c.

Observe that G is 2-connected but not 3-connected, since removing the vertices a and b disconnects G. One can check also that the equality $\delta(G) = \delta(G_1) + \delta(G_2) - 1$ holds. When we let $G_2 = K_3$, this construction is equivalent to trisecting the edge $\{a, b\}$ of G_1 and leaves the defect unchanged (that is, $\delta(G) = \delta(G_1)$). Plummer (1967) first studied this construction when G_1 is a complete graph and later Wessel (1970a) extended its work by showing that, in the above construction, the graph G must also be stability-critical and moreover that every connected non-3-connected stability-critical graph distinct from K_2 arises in this way.

We conclude this section by mentioning other references concerning stability-critical graphs: Beineke et al. (1967); Erdős et al. (1964); Harary and Plummer (1967); Sewell and Trotter (1995); Surányi (1975a, 1978, 1980); Wessel (1968, 1975, 1978, 1970b); Zhu (1989).

7 Facet-defining Graphs

Now considering weighted graphs as in Section 3, we will generalize stability-critical graphs. We assume from now on that all graphs we consider have at least three vertices. By Koppen's result (Proposition 3), all connected stability-critical graphs (with a constant weight 1 on all vertices) are such that their graphical inequality defines a facet of the linear ordering polytope P_{LO}^{Z}. Remember that the weighted graphs for which the graphical inequality defines a facet of P_{LO}^{Z} are the facet-defining graphs, or FDGs for short. Corollary 9 states that exactly the same weighted graphs produce a facet-defining inequality of the biorder polytope $P_{\mathrm{Bio}}^{X\times Y}$ (with some relationships between Z and X, Y). In this section we recall facts about FDGs obtained in Christophe et al. (2004) and describe a first set of new results. The next section provides more contributions about these graphs.

Let (G,μ) be a weighted graph. From Proposition 4 in Section 4 and Corollary 9 in Section 5, we derive exactly when (G,μ) is facet-defining in terms of the system $\mathcal{T}\cdot\mathcal{Y}=\mathcal{A}$ of (G,μ). Here is a reformulation of the condition.

Corollary 10 *A weighted graph (G,μ) is facet-defining if and only if for each nonzero valuation $\lambda : V(G)\cup E(G)\to\mathbb{Z}$ there is a tight set T of (G,μ) with*

$$\sum_{v\in T}\lambda(t)+\sum_{e\in E(T)}\lambda(e)\neq 0. \tag{16}$$

PROOF. By Proposition 4 and Corollary 9, (G,μ) is facet-defining if and only if the system $\mathcal{T}\cdot\mathcal{Y}=\mathcal{A}$ has only one solution. The latter condition amounts to: the homogeneous system $\mathcal{T}\cdot\mathcal{Y}=0$ has only the zero solution. In turn, this is equivalent to: the only linear combination of column vectors of $\mathcal{T}$ which produces the zero vector has only null coefficients. By contraposing, we get the claim. □

Corollary 10 is useful to derive necessary conditions for a weighted graph to be facet-defining, as illustrated in the next proposition.

Proposition 11 *Let (G,μ) be a FDG. Then the following conditions hold:*

(C1) *G is 2-connected;*
(C2) *for all $v\in V(G)$, we have $1\leq\mu(v)\leq\deg(v)-1$;*
(C3) *for all $\{v,w\}\in E(G)$, there is a tight set containing v and w;*
(C4) *for all $\{v,w\}\in E(G)$, there is a tight set containing neither v nor w;*
(C5) *for all $\{v,w\}\in E(G)$, there is a tight set containing v and not w;*

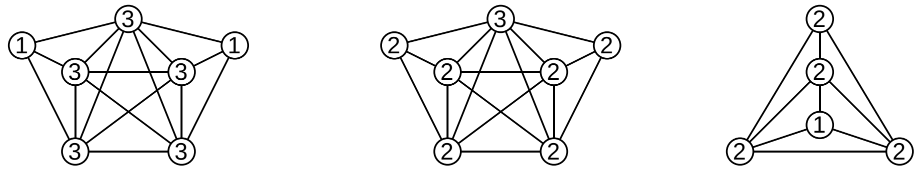

Fig. 2. Three specific FDGs.

(C6) *for all* $v, w \in V(G), \{v, w\} \notin E(G)$, *there is a tight set containing either both vertices* v *and* w *or none of them;*
(C7) *for all* $v, w \in V(G), \{v, w\} \notin E(G)$, *there is a tight set containing exactly one vertex of* $\{v, w\}$.

PROOF. (C1)–(C4) were already proved in Christophe et al. (2004) and we refer to it for (C1) and (C2). We prove (C3)–(C7) by using Corollary 10 with an appropriate choice for the valuation λ (this is a new proof for (C3)–(C4)).

(C3) Set $\lambda(\{v, w\}) = 1$ and λ to zero elsewhere.
(C4) Set $\lambda(v) = \mu(v) - \alpha(G, \mu)$, $\lambda(w) = \mu(w) - \alpha(G, \mu)$, $\lambda(\{v, w\}) = \alpha(G, \mu) - 1$, $\lambda(u) = \mu(u)$ for every $u \in V(G) \setminus \{v, w\}$ and $\lambda(e) = -1$ for every $e \in E(G) \setminus \{\{v, w\}\}$.
(C5) Set $\lambda(v) = 1$, $\lambda(\{v, w\}) = -1$ and λ to zero elsewhere.
(C6) Set $\lambda(v) = \mu(v) - \alpha(G, \mu)$, $\lambda(w) = \mu(w) - \alpha(G, \mu)$, $\lambda(u) = \mu(u)$ for every $u \in V(G) \setminus \{v, w\}$ and $\lambda(e) = -1$ for every $e \in E(G)$.
(C7) Set $\lambda(v) = 1$, $\lambda(w) = -1$ and λ to zero elsewhere.

It can easily be checked that each time the specific valuation λ ensures by Corollary 10 the existence of a tight set with the desired property. $\square$

There exist FDGs showing that Conditions (C6) and (C7) of Proposition 11 cannot be strengthened as in (C3), (C4) and (C5). Examples are given in Figure 2: in the left graph there is no tight set including the two vertices with unit weight, in the central one there is no tight set avoiding the two vertices with weight 2 and degree 3, and in the right one there is no tight set containing the unit weight vertex and not the only vertex nonadjacent to it.

Proposition 3 states that all stability-critical graphs together with the weight function $\mathbb{1}$ are facet-defining graphs (remember our present assumption that graphs have at least three vertices). Many more FDGs are derived by applying together Corollary 9 and techniques of Christophe et al. (2004), as we now explain. Let (G, μ) be a connected weighted graph. We say that (G, μ) is a *special facet-defining graph*, abbreviated SFDG, if for each $v \in V(G)$ we have $1 \leq \mu(v) \leq \deg(v) - 1$ and for each $v, w_1, \ldots, w_k \in V(G)$ such that $k = \mu(v)$ and $vw_1, \ldots, vw_k \in E(G)$, there exists a tight set T of (G, μ) containing $v, w_1, \ldots, w_k$. These graphs are all FDGs, as shown by the following proposition.

Proposition 12 (Christophe et al., 2004) *A SFDG is facet-defining, that is, any SFDG is a FDG.*

We note that in particular connected stability-critical graphs other than K_2 equipped with the weight function $\mathbb{1}$ are SFDG by Proposition 11(C3) and by Proposition 3. There exist FDGs which are not SFDGs, the three graphs in Figure 2 for instance.

We complete this section by stating an interesting result from Christophe et al. (2004) linking a weighted graph (G, μ) to the one obtained by taking $\deg - \mu$ as weight function, where $\deg : V(G) \to \mathbb{Z}$ assigns to each vertex its degree. Recall that we have $||G|| = |E(G)|$.

Proposition 13 (Christophe et al., 2004) *Let (G, μ) be a weighted graph. Then the following holds:*

(1) $\alpha(G, \deg - \mu) = \alpha(G, \mu) - \Big(\mu(V(G)) - ||G||\Big)$,
(2) a set $T \subseteq V(G)$ is tight in (G, μ) if and only if $V(G) \setminus T$ is tight in $(G, \deg - \mu)$, and
(3) (G, μ) is facet-defining if and only if $(G, \deg - \mu)$ is facet-defining.

In order to illustrate Proposition 13, we remark that the central graph in Figure 2 is obtained from the left one using the described modification of the weights.

8 The Defect of Facet-Defining Graphs

The defect of a graph G was defined in Section 6 as $\delta(G) = |V(G)| - 2\,\alpha(G)$. We generalize the concept to weighted graphs by letting the *defect* of (G, μ) be

$$\delta(G, \mu) = \mu(V(G)) - 2\,\alpha(G, \mu) \tag{17}$$

(when $\mu = \mathbb{1}$, we have $\delta(G, \mu) = \delta(G)$). We first observe a useful fact.

Proposition 14 *Let (G, μ) be a weighted graph. Then $\delta(G, \mu) = \delta(G, \deg - \mu)$.*

PROOF. The latter equality results from Proposition 13 in view of the following computations:

$$\begin{aligned}\delta(G,\mu) &= \mu(V(G)) - 2\alpha(G,\mu)\\ &= \mu(V(G)) - 2\Big(\alpha(G,\deg-\mu) + \mu(V(G)) - ||G||\Big)\\ &= 2||G|| - \mu(V(G)) - 2\alpha(G,\deg-\mu)\\ &= \deg(V(G)) - \mu(V(G)) - 2\alpha(G,\deg-\mu)\\ &= \delta(G,\deg-\mu). \qquad \square\end{aligned}$$

In order to prove lower bounds on the defect of a FDG (G,μ) we consider an enumeration of all its tight sets, say, $T_1, T_2, \ldots, T_k$. This enables us to explore the graph in the following way. We start from the empty graph and add the tight sets $T_1, T_2, \ldots, T_k$, one at a time. Clearly, the defect of (G,μ) equals the total weight of G minus the worth of T_1 and the worth of T_2. We first use some form of the inclusion-exclusion principle to write $\mu(V(G))$ as the weights of T_1 and T_2 plus the sum for i between 3 and k of the weight of T_i minus the weight of a certain vertex set denoted by X_i. By using the fact that the weight of any vertex set equals its worth plus the number of edges with both endpoints included in it, we obtain an expression for the defect which underlies our approach. Then, by analyzing the way the edges are discovered during the exploration of the graph, we derive lower bounds on $\delta(G,\mu)$.

For a sequence $\mathcal{T} = (T_1, T_2, \ldots, T_k)$ of k sets of vertices in a weighted graph (G,μ), we introduce $3\,(k-2)$ sets, for $3 \le j \le k$:

$$B_j^{\mathcal{T}} = (\cup_{h=1}^{j-1} T_h) \cap T_j, \qquad C_j^{\mathcal{T}} = (\cap_{h=1}^{j-1} T_h) \setminus T_j, \tag{18}$$

and

$$X_j^{\mathcal{T}} = B_j^{\mathcal{T}} \cup C_j^{\mathcal{T}}. \tag{19}$$

We will simply write B_j, C_j and X_j when the corresponding sequence $\mathcal{T}$ is clear from the context. The sets B_j and C_j are disjoint. Moreover, we have $C_i \cap C_j = \varnothing$ for $3 \le i \ne j \le k$.

Lemma 15 *Let (G,μ) be a weighted graph and $\mathcal{T} = (T_1, T_2, \ldots, T_k)$ be a sequence of subsets of $V(G)$ with $k \ge 2$. Then*

$$\mu(\cup_{i=1}^{k} T_i) + \mu(\cap_{i=1}^{k} T_i) = \sum_{i=1}^{k} \mu(T_i) - \sum_{j=3}^{k} \mu(X_j). \tag{20}$$

PROOF. For $k \ge 1$, we let

$$S_k = \mu(\cup_{i=1}^{k} T_i) + \mu(\cap_{i=1}^{k} T_i).$$

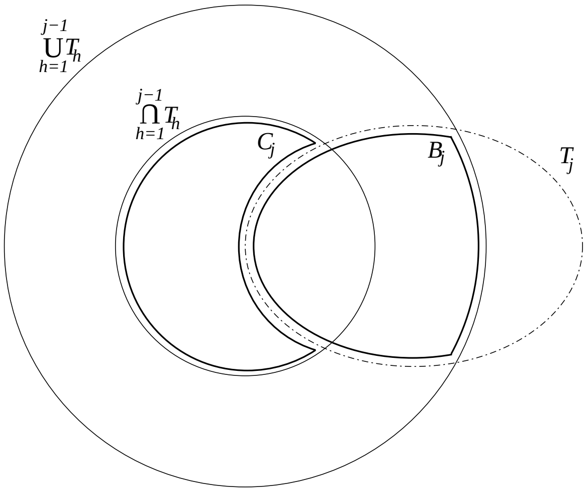

Fig. 3. The sets B_j and C_j.

Then

$$S_1 = \mu(T_1) + \mu(T_1), \qquad S_2 = \mu(T_1) + \mu(T_2), \tag{21}$$

and for $j \geq 3$, as illustrated in Figure 3, we have

$$\begin{aligned} S_j - S_{j-1} &= \mu(\cup_{h=1}^{j} T_h) - \mu(\cup_{h=1}^{j-1} T_h) + \mu(\cap_{h=1}^{j} T_h) - \mu(\cap_{h=1}^{j-1} T_h) \\ &= \mu(T_j) + \mu\Big((\cup_{h=1}^{j-1} T_h) \setminus T_j\Big) - \mu(\cup_{h=1}^{j-1} T_h) - \Big(\mu(\cap_{h=1}^{j-1} T_h) - \mu(\cap_{h=1}^{j} T_h)\Big) \\ &= \mu(T_j) - \mu\Big((\cup_{h=1}^{j-1} T_h) \cap T_j\Big) - \mu\Big((\cap_{h=1}^{j-1} T_h) \setminus T_j\Big) \\ &= \mu(T_j) - \mu(B_j \cup C_j) \\ &= \mu(T_j) - \mu(X_j). \end{aligned} \tag{22}$$

Equation (20) follows from Equations (21)–(22). □

For a sequence $\mathcal{T} = (T_1, T_2, \ldots, T_k)$ of tight sets of a weighted graph (G, μ), we will need to count separately the edges in the T_i's and in the X_j's. To this aim, we define the 'disjoint unions' of the respective collections of edges:

$$\mathfrak{T}_{\mathcal{T}} = \cup_{i=1}^{k} \{(e, i) \mid e \in E(T_i)\} = \cup_{i=1}^{k} \Big(E(T_i) \times \{i\}\Big), \tag{23}$$

and

$$\mathfrak{X}_{\mathcal{T}} = \cup_{j=3}^{k} \{(e, j) \mid e \in E(X_j)\} = \cup_{j=3}^{k} \Big(E(X_j) \times \{j\}\Big). \tag{24}$$

As for $X_j^{\mathcal{T}} = X_j$, we simply write $\mathfrak{X}$ and $\mathfrak{T}$ when the corresponding sequence $\mathcal{T}$ is clear from the context. The total number of tight sets of the weighted graph (G, μ) under consideration will be denoted as s. A *scenario* of (G, μ) is a list $(T_1, T_2, \ldots, T_s)$ of all tight sets of (G, μ).

Lemma 16 *Let (G, μ) be a FDG and $\mathcal{T} = (T_1, T_2, \ldots, T_s)$ be a scenario of*

(G, μ). *Then*

$$\delta(G, \mu) = |\mathfrak{T}| - |\mathfrak{X}| + \sum_{j=3}^{s} (w(T_j) - w(X_j)). \tag{25}$$

Remember that $||S||$ denotes the number of edges contained in the set S of vertices.

PROOF. By Conditions (C3) and (C4) of Proposition 11 and the fact that G is connected, we have $\cup_{i=1}^{s} T_i = V(G)$ and $\cap_{i=1}^{s} T_i = \varnothing$. Lemma 15 then gives

$$\begin{aligned}
\mu(V(G)) &= \mu(\cup_{i=1}^{s} T_i) + \mu(\cap_{i=1}^{s} T_i) \\
&= \sum_{i=1}^{s} \mu(T_i) - \sum_{j=3}^{s} \mu(X_j) \\
&= w(T_1) + ||T_1|| + w(T_2) + ||T_2|| + \\
&\quad \sum_{j=3}^{s} \Big(w(T_j) + ||T_j|| \Big) - \sum_{j=3}^{s} \Big(w(X_j) + ||X_j|| \Big) \\
&= 2\alpha(G, \mu) + \sum_{i=1}^{s} ||T_i|| - \sum_{j=3}^{s} ||X_j|| + \sum_{j=3}^{s} \Big(w(T_j) - w(X_j) \Big) \\
&= 2\alpha(G, \mu) + |\mathfrak{T}| - |\mathfrak{X}| + \sum_{j=3}^{s} \Big(w(T_j) - w(X_j) \Big). \qquad \square
\end{aligned}$$

Building upon the previous lemma, we now show the positivity of the defect of a FDG.

Proposition 17 *The defect of any FDG (G, μ) satisfies $\delta(G, \mu) \geq 1$.*

PROOF. Taking again any scenario $\mathcal{T} = (T_1, T_2, \ldots, T_s)$ of (G, μ), we refer to Equation (25) in Lemma 16. Because T_j is assumed to be a tight set, we have $w(T_j) - w(X_j) \geq 0$. To prove $\delta(G, \mu) \geq 1$, it suffices to show $|\mathfrak{X}| \leq |\mathfrak{T}| + 1$. We first exhibit an injective mapping φ from $\mathfrak{X}$ to $\mathfrak{T}$, built for any given scenario $\mathcal{T}$. As will be seen, $\varphi(e, j) = (e, j')$ for some j' in all cases. Then for an appropriate choice of the scenario, we show the existence of an element in $\mathfrak{T} \setminus \varphi(\mathfrak{X})$.

Let $(e, j) \in \mathfrak{X}$, for some j in $\{3, 4, \ldots, s\}$, where $e = \{u, v\}$. Thus $e \in E(X_j)$ with $X_j = B_j \cup C_j$. This leads to three cases.

Case 1. Assume $e \in E(B_j)$. Because $B_j \subseteq T_j$, we have $(e, j) \in E(T_j) \times \{j\}$. We then set $\varphi(e, j) = (e, j)$. Here is an illustration: the symbol $*$ indicates where we select $\varphi(e, j)$ (blank entries are undetermined and can be filled arbitrarily with $\in$ or $\notin$).

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	$\ldots$	T_j
u					$\in$				$\in$
v			$\in$						$\in$
									$*$

Clearly, distinct pairs (e, j) satisfying $e \in B_j$ have distinct images.

Case 2. Assume $e \in E(C_j)$. Because of the definition of C_j together with $j \geq 3$, we get $e \in E(T_1)$. We then set $\varphi(e, j) = (e, 1)$.

	T_1	T_2	T_3	$\ldots$	T_{j-1}	T_j
u	$\in$	$\in$	$\in$	$\ldots$	$\in$	$\notin$
v	$\in$	$\in$	$\in$	$\ldots$	$\in$	$\notin$
	$*$					

For each vertex w of e, the index j is the least value such that $j \geq 3$ and $w \notin T_j$. Consequently, pairs (e, j) with $e \in E(C_j)$ have distinct images by φ, and all those images differ from the Case 1 images.

Case 3. Assume now $u \in B_j$ and $v \in C_j$. Exchanging u and v if necessary, this is the last possible case. Again, j is well defined from e. We consider subcases according to the value c of the least index l such that $u \notin T_l$ (necessarily $c \neq j$ because $u \in B_j \subseteq T_j$).

Subcase 3.1. When $c = 1$, we take $r = \min\{h \mid e \in E(T_h)\}$, and set $\varphi(e, j) = (e, r)$.

	T_1	T_2	$\ldots$	T_{r-1}	T_r	$\ldots$	T_{j-1}	T_j
u	$\notin$	$\notin$	$\ldots$	$\notin$	$\in$			$\in$
v	$\in$	$\in$	$\ldots$	$\in$	$\in$	$\ldots$	$\in$	$\notin$
					$*$			

Because $1 < r$ and $e \notin E(B_r)$, we conclude that all of these images are distinct and moreover differ from the images obtained in Cases 1 and 2.

Subcase 3.2. When $2 \leq c < j$, we set $\varphi(e, j) = (e, 1)$.

	T_1	T_2	$\ldots$	T_{c-1}	T_c	$\ldots$	T_{j-1}	T_j
u	$\in$	$\in$	$\ldots$	$\in$	$\notin$			$\in$
v	$\in$	$\in$	$\ldots$	$\in$	$\in$	$\ldots$	$\in$	$\notin$
	$*$							

Distinct pairs falling in this case have distinct images by φ. For a fixed edge e, there cannot exist two distinct pairs (e, j) in $\mathfrak{X}$ such that one falls in Case 2 and the other one in Case 3, so images from the actual subcase differ from those obtained in all preceding cases.

Subcase 3.3. The only remaining case is when $j < c$. We then set $\varphi(e, j) = (e, 2)$.

$$\begin{array}{ccccccccc}
 & T_1 & T_2 & \dots & T_{j-1} & T_j & \dots & T_{c-1} & T_c \\
u & \in & \in & \dots & \in & \in & \dots & \in & \notin \\
v & \in & \in & \dots & \in & \notin & & & \\
 & & * & & & & & &
\end{array}$$

Again, distinct pairs in this case have distinct images, and as it is the only case where the pair $(e, 2)$ can be selected, the injectivity of the resulting mapping $\varphi : \mathfrak{X} \to \mathfrak{T}$ holds for any given scenario $\mathcal{T}$.

Now, let $e \in E(G)$ and choose tight sets T_1, T_2 such that $e \subseteq T_1$ and $e \cap T_2 = \varnothing$ (these tight sets must exist by Proposition 11(C3) and (C4)). Take any scenario starting with T_1 and T_2. Noticing $(e, 1) \in \mathfrak{T} \setminus \varphi(\mathfrak{X})$, we infer $\delta(G, \mu) \geq 1$. $\square$

By Proposition 11, Condition (C2), we have $\deg(v) - \mu(v) \geq 1$. Thus the following result strengthens Proposition 17.

Proposition 18 *Let (G, μ) be a FDG and v be any of its vertices. Then $\delta(G, \mu) \geq \deg(v) - \mu(v)$.*

In the particular case of stability-critical graphs, Proposition 18 gives $\deg(v) \leq \delta(G) + 1$, a theorem of Hajnal (1965) (recalled in Section 6).

PROOF. Let $\mathcal{T}$ be a scenario of (G, μ) in which the tight sets containing v are listed before those not containing v. Let T_l be the first tight set of $\mathcal{T}$ which does not contain v. The set $N(v)$ of neighbors of v is partitioned into the three following subsets:

$$\begin{aligned}
X &= \big(N(v) \cap T_1\big) \setminus T_l, \\
Y &= N(v) \setminus (T_1 \cup T_l), \\
Z &= N(v) \cap T_l.
\end{aligned}$$

Consider the injective mapping $\varphi : \mathfrak{X} \to \mathfrak{T}$ built in the proof of Proposition 17 for the scenario $\mathcal{T}$. Then for all $x \in X$ we get either $(\{v, x\}, 1) \in \mathfrak{T} \setminus \varphi(\mathfrak{X})$, in case $x \notin T_2$, or $(\{v, x\}, 2) \in \mathfrak{T} \setminus \varphi(\mathfrak{X})$, in case $x \in T_2$. Also, for all $y \in Y$, we have $(\{v, y\}, r) \in \mathfrak{T} \setminus \varphi(\mathfrak{X})$, where $r = \min\{h \mid \{vy\} \in E(T_h)\}$. Consequently, $|\mathfrak{T}| - |\mathfrak{X}| \geq |X \cup Y| = \deg(v) - |Z|$.

Moreover, we observe $v \notin T_l$, $v \in X_l$ and $B_l \cap N(v) = Z$, $C_l \cap N(v) = \varnothing$. It follows, because T_l is tight, $w(T_l) - w(X_l) \geq w(X_l \setminus \{v\}) - w(X_l) = |Z| - \mu(v)$.

Combining these two observations with Lemma 16 yields (remember that s is the total number of tight sets)

$$\begin{aligned}\delta(G,\mu) &= |\mathfrak{T}| - |\mathfrak{X}| + \sum_{j=3}^{s} \Big(w(T_j) - w(X_j)\Big) \\ &\geq |\mathfrak{T}| - |\mathfrak{X}| + w(T_l) - w(X_l) \\ &\geq \deg(v) - |Z| + |Z| - \mu(v) \\ &= \deg(v) - \mu(v). \qquad \square\end{aligned}$$

Corollary 19 *Let (G,μ) be a FDG and v be any of its vertices. Then $\mu(v) \leq \delta(G,\mu)$.*

PROOF. By Proposition 13, $(G, \deg - \mu)$ is also a FDG and by Proposition 14, $\delta(G,\mu) = \delta(G, \deg - \mu)$. Applying Proposition 18 to $(G, \deg - \mu)$ and v gives the claim. $\square$

Combining Proposition 18 and Corollary 19 gives an upper bound of $2\delta(G,\mu)$ for the degree of a vertex in a FDG (G,μ). Recently, one of us (G.J.) established the stronger bound $\deg(v) \leq 2\delta(G,\mu) - 1$ when $\delta(G,\mu) \geq 2$ (this is the best possible bound: for each natural number $d \geq 2$, there are examples of FDGs (G,μ) with a vertex v such that $\deg(v) = 2\delta(G,\mu) - 1$ and $\delta(G,\mu) = d$). When a vertex of a FDG satisfies a certain technical condition, we are able to prove an even stronger result.

Proposition 20 *Let (G,μ) be a FDG and $v \in V(G)$ one of its vertices. Suppose that for every set S of $\mu(v)$ neighbors of v there exists a tight set T such that $S \cup \{v\} \subseteq T$. Then $\deg(v) \leq \delta(G,\mu) + 1$.*

For instance, the above assumption is fulfilled for every vertex when (G,μ) is any SFDG (in the sense of Section 7).

PROOF. Let $k = \mu(v)$ and $l = \deg(v) - \mu(v)$. By Proposition 11(C2), we have $l \geq 1$. When $k = 1$ the claim follows from Proposition 18, so we assume $k \geq 2$. Let $W = \{w_1, w_2, \ldots, w_{k+l}\}$ be the set of neighbors of v. Let also $\mathcal{T} = (T_1, \ldots, T_s)$ be a scenario of (G,μ) such that the first $l + k + 1$ tight sets are specified as follows. For $1 \leq i \leq l + 1$, the tight set T_i contains v, w_1, w_2, $\ldots$, w_{k-1}, w_{k+i-1}. For $l + 2 \leq j \leq l + k$, the tight set T_j contains v, w_1, w_2, $\ldots$, w_{j-l-2}, w_{j-l}, w_{j-l+1}, $\ldots$, w_{k+1}. Finally, we let T_{l+k+1} be a tight set such that $\{w_1, w_2, \ldots, w_k\} \subseteq T_{l+k+1}$ and $v \notin T_{l+k+1}$. All these tight sets exist by hypothesis, and they all contain exactly k neighbors of the vertex v.

Let now $\varphi : \mathfrak{X} \to \mathfrak{T}$ be the injective mapping defined for the scenario $\mathcal{T}$ as in the proof of Proposition 17. Then $(\{v, w_{k+i}\}, i+1) \in \mathfrak{T} \setminus \varphi(\mathfrak{X})$ can be checked for $1 \leq i \leq l$. Also, for $1 \leq j \leq k$, we have $w_j \in C_{l+j+1}$ and v, w_1, w_2, $\ldots$, w_{j-1}, w_{j+1}, $\ldots$, $w_{k+1} \in B_{l+j+1}$, giving $w(X_{l+j+1} \setminus \{v\}) - w(X_{l+j+1}) \geq 1$. Using Lemma 16 we then deduce

$$\begin{aligned}
\delta(G,\mu) &= |\mathfrak{T}| - |\mathfrak{X}| + \sum_{j=3}^{s} \Big(w(T_j) - w(X_j) \Big) \\
&\geq |\mathfrak{T}| - |\mathfrak{X}| + \sum_{j=l+2}^{l+k} \Big(w(T_j) - w(X_j) \Big) \\
&\geq |\mathfrak{T}| - |\mathfrak{X}| + \sum_{j=l+2}^{l+k} \Big(w(X_j \setminus \{v\}) - w(X_j) \Big) \\
&\geq |\mathfrak{T}| - |\mathfrak{X}| + k - 1 \\
&\geq l + k - 1 = \deg(v) - 1. \qquad \square
\end{aligned}$$

The defect was shown to be a useful invariant for the investigation of stability-critical graphs, in particular for attempting to classify these graphs (see Section 6). In view of the current section, the same assertion applies also to the weighted case. We now make a first elementary step in the classification of FDGs.

Proposition 21 *FDGs of defect one are the odd cycles with the weight function* $\mathbb{1}$.

PROOF. Let (G, μ) be a FDG of defect one. By Proposition 11(C2) and Corollary 19, it follows that $\mu = \mathbb{1}$. By Proposition 3, the graph G must be a stability-critical graph of defect one, that is, an odd cycle. $\square$

One of the authors (G.J.) is currently investigating FDGs of defect 2. As it is the case for stability-critical graphs (see Section 6), all such FDGs are generated by repeatedly applying some well defined construction to a finite number of basic graphs. Whether a similar result holds for all FDGs of any fixed defect is a question left for future work.

9 Conclusion

The whole family of fence inequalities for the linear ordering polytope have been subsumed to a general form of facet-defining inequalities. Called the graphical inequalities, the latter are built from specific weighted graphs. The weighted graphs which define facets in this way generalize stability-critical graphs. They are investigated, in particular with regard to their defect. We point out that Corollary 9 and Proposition 13 imply that connected stability-critical graphs produce a facet not only as in Koppen (1995) (that is, taken with all weights equal to 1), but also when the weight of any vertex v is set to the degree of v minus 1.

We mention that there are facet-defining inequalities for the linear ordering polytope which are not graphical, for instance the Möbius inequalities: see, e.g., Grötschel et al. (1985a), Borndörfer and Weismantel (2000), and Fiorini (2006a,b).

References

Andrásfai, B., 1967. On critical graphs. In: Theory of Graphs (Internat. Sympos., Rome, 1966). Gordon and Breach, New York, pp. 9–19.

Beineke, L. W., Harary, F., Plummer, M., 1967. On the critical lines of a graph. Pacific Journal of Mathematics 22, 205–212.

Block, H. D., Marschak, J., 1960. Random orderings and stochastic theories of responses. In: Olkin, I., Ghurye, S., Hoeffding, W., Madow, W., Mann, H. (Eds.), Contributions to probability and statistics. Stanford Univ. Press, Stanford, Calif., pp. 97–132.

Bolotashvili, G., 1987. A class of facets of the permutation polytope and a method for constructing facets of the permutation polytope. Tech. Rep. 3403-B87 and N3405-B87, VINITI (in Russian).

Bolotashvili, G., Kovalev, M., Girlich, E., 1999. New facets of the linear ordering polytope. SIAM Journal on Discrete Mathematics 12, 326–336.

Borndörfer, R., Weismantel, R., 2000. Set packing relaxations of some integer programs. Mathematical Programming 88 (Ser. A), 425–450.

Christophe, J., Doignon, J.-P., Fiorini, S., 2003. Counting biorders. Journal of Integer Sequences 6, article 03.4.3, 10 pages.

Christophe, J., Doignon, J.-P., Fiorini, S., 2004. The biorder polytope. Order 21, 61–82.

Cohen, M., Falmagne, J.-C., 1978. Random scale representation of binary choice probabilities: A counterexample to a conjecture of Marschak. Tech. rep., Department of Psychology, New York University, New York, unpublished manuscript.

Cohen, M., Falmagne, J.-C., 1990. Random utility representation of binary choice probabilities: a new class of necessary conditions. Journal of Mathematical Psychology 34, 88–94.

Diestel, R., 2005. Graph theory, 3rd Edition. Vol. 173 of Graduate Texts in Mathematics. Springer-Verlag, Berlin.

Doignon, J.-P., Ducamp, A., Falmagne, J.-C., 1984. On realizable biorders and the biorder dimension of a relation. Journal of Mathematical Psychology 28, 73–109.

Ducamp, A., Falmagne, J. C., 1969. Composite measurement. Journal of Mathematical Psychology 6, 359–390.

Erdős, P., Gallai, T., 1961. On the minimal number of vertices representing the edges of a graph. A Magyar Tudományos Akadémia Matematikai és Fizikai Tudományok Osztályának Közleményei 6, 181–203.

Erdős, P., Hajnal, A., Moon, J. W., 1964. A problem in graph theory. The American Mathematical Monthly 71, 1107–1110.

Falmagne, J.-C., 1978. A representation theorem for finite random scale systems. Journal of Mathematical Psychology 18, 52–72.

Falmagne, J.-C., 1979. Errata: "A representation theorem for finite random scale systems" (J. Math. Psych. **18** (1978), no.1, 52–72). Journal of Mathematical Psychology 19, 219.

Fiorini, S., 2001. Determining the automorphism group of the linear ordering polytope. Discrete Applied Mathematics 112, 121–128.

Fiorini, S., 2004. A short proof of a theorem of Falmagne. Journal of Mathematical Psychology 48, 80–82.

Fiorini, S., 2006a. How to recycle your facets, accepted for publication in *Discrete Optimization*.

Fiorini, S., 2006b. $\{0, \frac{1}{2}\}$-cuts and the linear ordering problem: surfaces that define facets, conditionally accepted for publication in *SIAM Journal on Discrete Mathematics*.

Fishburn, P. C., 1990. Binary probabilities induced by rankings. SIAM Journal on Discrete Mathematics 3, 478–488.

Gilboa, I., 1990. A necessary but insufficient condition for the stochastic binary choice problem. Journal of Mathematical Psychology 34, 371–392.

Grötschel, M., Jünger, M., Reinelt, G., 1985a. Facets of the linear ordering polytope. Mathematical Programming 33, 43–60.

Grötschel, M., Jünger, M., Reinelt, G., 1985b. On the acyclic subgraph polytope. Mathematical Programming 33, 28–42.

Guilbaud, G., 1953. Sur une difficulté de la théorie du risque. In: Econométrie. Vol. 1952 of Colloques Internationaux du Centre National de la Recherche Scientifique, no. 40, Paris. Centre de la Recherche Scientifique, Paris, pp. 19–25; discussion, pp. 25–28.

Guttman, L., 1944. A basis for scaling quantitative data. American Sociological Review 9, 139–150.

Hajnal, A., 1965. A theorem on k-saturated graphs. Canadian Journal of Mathematics 17, 720–724.

Harary, F., Plummer, M. D., 1967. On indecomposable graphs. Canadian Jour-

nal of Mathematics 19, 800–809.
Karp, R. M., 1972. Reducibility among combinatorial problems. In: Complexity of computer computations (Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1972). Plenum, New York, pp. 85–103.
Koppen, M., 1991. Random utility representation of binary choice probabilities. In: Doignon, J.-P., Falmagne, J.-C. (Eds.), Mathematical psychology: current developments. Springer Verlag, New York, pp. 181–201.
Koppen, M., 1995. Random utility representation of binary choice probabilities: critical graphs yielding critical necessary conditions. Journal of Mathematical Psychology 39, 21–39.
Leung, J., Lee, J., 1994. More facets from fences for linear ordering and acyclic subgraph polytopes. Discrete Applied Mathematics 50, 185–200.
Lipták, L., Lovász, L., 2000. Facets with fixed defect of the stable set polytope. Mathematical Programming 88 (Ser. A), 33–44.
Lipták, L., Lovász, L., 2001. Critical facets of the stable set polytope. Combinatorica 21, 61–88.
Lovász, L., 1978. Some finite basis theorems on graph theory. In: Hajnal, A., Sós, V. (Eds.), Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely,1976), Vol. II. Vol. 18 of Colloquia Mathematica Societatis János Bolyai. North-Holland, Amsterdam, pp. 717–729.
Lovász, L., 1993. Combinatorial Problems and Exercises, 2nd Edition. North-Holland Publishing Co., Amsterdam.
Lovász, L., Plummer, M. D., 1986. Matching Theory. Vol. 121 of North-Holland Mathematics Studies. North-Holland Publishing Co., Amsterdam, Annals of Discrete Mathematics, 29.
McLennan, A., 1990. Binary stochastic choice. In: Chipman, J., McFadden, D., Richter, M. (Eds.), Preferences, Uncertainty, and Optimality. Westview Press, Boulder, CO, pp. 187–202.
Megiddo, N., 1977. Mixtures of order matrices and generalized order matrices. Discrete Mathematics 19, 177–181.
Plummer, M. D., 1967. On a family of line-critical graphs. Monatshefte für Mathematik 71, 40–48.
Riguet, J., 1951. Les relations de Ferrers. Comptes Rendus des Séances de l'Académie des Sciences (Paris) 232, 1729–1730.
Sewell, E. C., Trotter, Jr., L. E., 1993. Stability critical graphs and even subdivisions of K_4. Journal of Combinatorial Theory. Series B 59, 74–84.
Sewell, E. C., Trotter, Jr., L. E., 1995. Stability critical graphs and ranks facets of the stable set polytope. Discrete Mathematics 147, 247–255.
Suck, R., 1992. Geometric and combinatorial properties of the polytope of binary choice probabilities. Mathematical Social Sciences 23, 81–102.
Surányi, L., 1975a. Large α-critical graphs with small deficiency. On line-critical graphs. II. Studia Scientiarum Mathematicarum Hungarica 10, 397–412 (1978).
Surányi, L., 1975b. On line critical graphs. In: Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. III.

North-Holland, Amsterdam, pp. 1411–1444. Colloquia Mathematica Societatis János Bolyai, Vol. 10.

Surányi, L., 1978. A note on a conjecture of Gallai concerning α-critical graphs. In: Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. II. Vol. 18 of Colloquia Mathematica Societatis János Bolyai. North-Holland, Amsterdam, pp. 1065–1074.

Surányi, L., 1980. On a generalization of line-critical graphs. Discrete Mathematics 30, 277–287.

Wessel, W., 1968. Eine Methode zur Konstruktion von kanten-p-kritischen Graphen. In: Sachs, H., Voss, H.-J., Walther, H. (Eds.), Beiträge zur Graphentheorie (Kolloquium, Manebach, 1967). Teubner, Leipzig, pp. 207–210.

Wessel, W., 1970a. Kanten-kritische Graphen mit der Zusammenhangszahl 2. Manuscripta Mathematica 2, 309–334.

Wessel, W., 1970b. On the problem of determining whether a given graph is edgecritical or not. In: Erdös, P., Rényi, A., Sós, V. (Eds.), Combinatorial theory and its applications, III (Proc. Colloq., Balatonfüred, 1969). North-Holland, Amsterdam, pp. 1123–1139.

Wessel, W., 1975. A first family of edge-critical wheels. Periodica Mathematica Hungarica 6, 229–233.

Wessel, W., 1978. A second family of edge-critical wheels. In: Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. II. Vol. 18 of Colloquia Mathematica Societatis János Bolyai. North-Holland, Amsterdam, pp. 1123–1145.

Zhu, Q. C., 1989. The structure of α-critical graphs with $|V(G)| - 2\alpha(G) = 3$. In: Graph theory and its applications: East and West (Jinan, 1986). Annals of the New York Academy of Sciences 576. New York Acad. Sci., New York, pp. 716–722.