Well-balanced orientations of mixed graphs

Attila Bernáth* Gwenaël Joret†

Abstract

We show that deciding if a mixed graph has a well-balanced orientation is NP-complete.

1 Introduction

Consider the following problem: given an undirected graph G, orient the edges of G in such a way that in the resulting directed graph $\vec{G}$, we have at least $\lfloor \lambda_G(x,y)/2 \rfloor$ directed edge-disjoint paths from x to y, for all $x, y \in V(G)$. Here, $\lambda_G(x,y)$ denotes the maximum number of edge-disjoint paths between x and y in G. Such an orientation of G is said to be a *well-balanced* orientation. An important theorem of Nash-Williams [5] asserts that every graph has a well-balanced orientation (see Frank [2] for a simpler proof). Also, as explained in [3], a well-balanced orientation of G can be found in polynomial time.

The above mentioned theorem of Nash-Williams on the existence of well-balanced orientations has intrigued many mathematicians since it was born (which was quite long ago, in 1960). The reason for this is that the theorem, the generalization of it which was in fact proved and the proof method itself is so different from other results and methods in graph theory that no connection with other areas has been found since then. We can say that not much more is known about the problem since 1960. In [4] many approaches were presented to obtain generalizations of this theorem, but with little success: most of the questions raised there were answered negatively with counter-examples, though some remained open. Here we decide one of these open problems.

Recently, several generalizations of the above problem were shown to be NP-complete by Bernáth [1]. For instance, if for every edge $\{x, y\} \in E(G)$ we are given non negative costs c_{xy}, c_{yx} for orienting the edge from x to y, and from y to x respectively, then deciding if G has a well-balanced orientation of cost at most a given bound K is NP-complete.

In this note, we are concerned with the following special case of the problem: we are given a graph G where some edges are already oriented (a *mixed* graph), and we want to decide if the remaining undirected edges can be oriented in such

*Dept. of Operations Research, Eötvös University, Pázmány P. s. 1/C, Budapest, Hungary H-1117. The author is a member of the Egerváry Research Group (EGRES). Research was supported by OTKA grants K60802 and TS 049788 and by the Egerváry Research Group of the Hungarian Academy of Sciences. E-mail: bernath@cs.elte.hu

†Computer Science Department, Université Libre de Bruxelles, CP 212, B-1050 Brussels, Belgium. Aspirant F.R.S.–FNRS. This research was done while this author visited the Egerváry Research Group (EGRES) at Eötvös University, supported by European MCRTN Adonet, Contract Grant No. 504438. E-mail: gjoret@ulb.ac.be

a way as to obtain a well-balanced orientation of the underlying undirected graph. The complexity of this problem was posed as an open question in [4]. We will show that it is also NP-complete:

Theorem 1. *Deciding whether a mixed graph has an orientation that is a well-balanced orientation of the underlying undirected graph is NP-complete.*

2 The Reduction

For the reduction we need a special form of the Vertex Cover problem:

Lemma 1. *Given a graph with $2n$ vertices and no isolated vertex, it is NP-complete to decide whether there exists a vertex cover of size at most n.*

Proof. It is well known that the Vertex Cover problem is NP-complete, we will reduce it to the above problem. Assume we are given an instance of the Vertex Cover problem consisting of a graph $G = (V, E)$ and a positive integer k where the question is whether G has a vertex cover of size at most k. We may clearly assume that G has no isolated vertex. Distinguish the following cases:

1. If $k = |V|/2$ then we are done.
2. If $k > |V|/2$ then let G' be the disjoint union of G and $K_{t,1}$ for $t = 2k + 1 - |V|$. Then G has a vertex cover of size at most k iff G' has a vertex cover of size at most $k + 1$. Since G' has $2k + 2$ vertices, G' is an instance of the problem in the lemma.
3. If $k < |V|/2$ then let G' be the disjoint union of G and K_t for $t = |V| + 2 - 2k$. Then G has a vertex cover of size at most k iff G' has a vertex cover of size at most $k + t - 1$. As G' has $2(k + t - 1)$ vertices, G' is again an instance of the problem in the lemma.

The reduction takes clearly polynomial time, hence the lemma follows. We note that G' has no isolated vertices. □

Let us introduce some notations. A mixed graph will be denoted by a triple (V, E, A) where V is the set of nodes, E is the set of undirected edges and A is the set of directed edges. For a given directed graph D and $x, y \in V(D)$, we denote by $\lambda_D(x, y)$ the maximum number of directed edge-disjoint paths from x to y in D. Also, for $S \subseteq V(D)$, we use $\varrho_D(S)$ for the number of arcs in D going from $V(D) - S$ to S (when $S = \{v\}$, we simply write $\varrho_D(v)$). We note that, by Menger's theorem, we have

$$\lambda_D(x, y) = \min_{\substack{S \subseteq V(D),\\ x \notin S, y \in S}} \varrho_D(S).$$

We may now turn to the reduction.

Proof of Theorem 1. The problem is easily seen to be in NP, so let us prove its completeness. To this end we will reduce the problem in Lemma 1 to our problem using a construction similar to those in [1]. So suppose we are given an instance $G' = (V', E')$ of the problem in Lemma 1. We remark that we wanted to

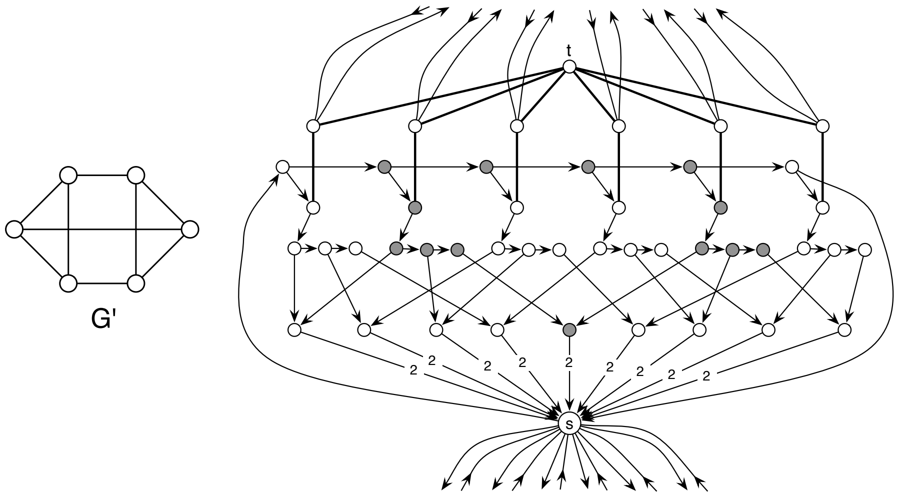

Figure 1: Illustration of the reduction.

avoid isolated vertices in G' only to make the following argumentation simpler. Consider the following mixed graph $M = (V, E, A)$: the vertex set V will contain two designated vertices s and t, $d_{G'}(v) + 3$ vertices $y^v, z^v, x_0^v, x_1^v, x_2^v, \ldots, x_{d_{G'}(v)}^v$ for every $v \in V'$, and one vertex x_e for every $e \in E'$. Let us fix an ordering of V', say $V' = \{v_1, v_2, \ldots, v_n\}$. The arc set A of M contains a directed circuit on $s, z^{v_1}, z^{v_2} \ldots, z^{v_n}$ in this order, a pair of oppositely directed arcs between s and y^v for every $v \in V'$, arcs (z^v, x_0^v) and (x_i^v, x_{i+1}^v) for $i = 0, \ldots, d_{G'}(v) - 1$ and every $v \in V'$, two parallel arcs from x_e to s for every $e \in E'$ and finally for each $v \in V'$ take an arbitrary order of the $d_{G'}(v) = d$ edges of G' incident to v, say $e^1, e^2, \ldots, e^d$, and include the arc (x_i^v, x_{e^i}) for every $i \in \{1, \ldots, d\}$.

The edge set E of M contains one edge between t and y^v and one edge between y^v and x_0^v for every $v \in V'$.

The construction is illustrated in Figure 1. The arcs with a label "2" indicate a multiplicity of 2; the undirected edges are drawn in bold.

Let G be the underlying undirected graph of M and $D = (V, A)$ be the directed part of M. Notice that $\lambda_G(x, y) = \min\{d_G(x), d_G(y)\}$ for every $x, y \in V$ (for example one can check that this is true if $y = s$ from which it follows for arbitrary x, y). Observe that $D - t$ is strongly connected and that $\lambda_D(x_e, s) = 2$ for each $e \in E'$.

Observe furthermore that the well-balanced orientations of M are necessarily of the following form: the two edges of E incident to a vertex y^v with $v \in V'$ form a directed path of length two, and for exactly half (i.e. $|V'|/2$) of these, this path starts at t, and for the other half this path ends at t. In other words, $\varrho_{\vec{M}}(y^v) = 2$ for all $v \in V'$ and $\varrho_{\vec{M}}(t) = |V'|/2$ in any well-balanced orientation $\vec{M}$ of M. This is implied by the edge-connectivities in G.

If G' has a vertex cover of size at most $|V'|/2$ then it has one, say S, of size exactly $|V'|/2$. By orienting for every $v \in V'$ the path t, y^v, x_0^v from left to right if $v \in S$, and from right to left otherwise, it is easily seen that we get a well-balanced orientation of M.

Suppose now that M admits a well-balanced orientation $\vec{M}$ and consider the set $S \subseteq V'$ of vertices of G' for which the corresponding directed paths in $\vec{M}$ start at t, that is

$$S := \{v \in V' : \ (t, y^v) \text{ and } (y^v, x_0^v) \text{ are arcs of } \vec{M}\}.$$

We claim that S forms a vertex cover of G': if edge $e = \{v_j, v_k\} \in E'$ were not covered by S (where $j < k$ are the indices of the vertices in the fixed ordering), then $\varrho_{\vec{M}}(X) = 1$ would contradict the well-balancedness of $\vec{M}$, where

$$\begin{aligned} X \quad := \quad & \{x_e\} \bigcup \{z^{v_i} : j \leq i \leq k\} \\ & \bigcup \{x_i^{v_j} : 0 \leq i \leq d_{G'}(v_j)\} \bigcup \{x_i^{v_k} : 0 \leq i \leq d_{G'}(v_k)\} \end{aligned}$$

(the vertices in grey in Figure 1 illustrate this cut). □

Acknowledgements

We are grateful to the anonymous referee for pointing out an error in an earlier version of this note.

References

[1] A. Bernáth, *Hardness results for well-balanced orientations*, Tech. Report TR-2006-05, Egerváry Research Group, Budapest, 2006, `www.cs.elte.hu/egres`.

[2] A. Frank, *Applications of submodular functions*, Surveys in combinatorics, 1993 (Keele), London Math. Soc. Lecture Note Ser., vol. 187, Cambridge Univ. Press, Cambridge, 1993, pp. 85–136.

[3] H. N. Gabow, *Efficient splitting off algorithms for graphs*, STOC '94: Proceedings of the twenty-sixth annual ACM symposium on Theory of computing (New York, NY, USA), ACM Press, 1994, pp. 696–705.

[4] S. Iwata, T. Király, Z. Király, and Z. Szigeti, *On well-balanced orientations, counter-examples for related problems*, Tech. Report TR-2004-16, Egerváry Research Group, Budapest, 2004, `www.cs.elte.hu/egres`.

[5] C. St. J. A. Nash-Williams, *On orientations, connectivity and odd-vertex-pairings in finite graphs.*, Canad. J. Math. **12** (1960), 555–567.