Turán's Theorem and k-Connected Graphs

Nicolas Bougard*
DÉPARTEMENT DE MATHÉMATIQUE
UNIVERSITÉ LIBRE DE BRUXELLES
BOULEVARD DU TRIOMPHE, C.P. 216
B-1050 BRUXELLES, BELGIUM

Gwenaël Joret†
DÉPARTEMENT D'INFORMATIQUE
UNIVERSITÉ LIBRE DE BRUXELLES
BOULEVARD DU TRIOMPHE, C.P. 212
B-1050 BRUXELLES, BELGIUM

ABSTRACT

The minimum size of a k-connected graph with given order and stability number is investigated. If no connectivity is required, the answer is given by Turán's Theorem. For connected graphs, the problem has been solved recently independently by Christophe et al., and by Gitler and Valencia. In this paper, we give a short proof of their result and determine the extremal graphs. We settle the case of 2-connected graphs, characterize the corresponding extremal graphs, and also extend a result of Brouwer related to Turán's Theorem.

Keywords: Turán's Theorem, α-Critical Graphs

* Nicolas Bougard is a Research Fellow of the Fonds pour la formation à la Recherche dans l'Industrie et dans l'Agriculture (FRIA).
† Gwenaël Joret is a Research Fellow of the Fonds National de la Recherche Scientifique (FNRS).

1. INTRODUCTION

In this paper, all graphs are supposed to be finite and simple (i.e. without loops and multiple edges). Let n, α be two positive integers such that $n \geq \alpha$. A classical result of Turán [11] shows that there is a unique graph with order n and stability number α such that the size (number of edges) is minimal: The *Turán graph* $T(n, \alpha)$, consisting of α disjoint balanced cliques.

The graph $T(n, \alpha)$ is not connected, except for $\alpha = 1$. Therefore, we investigate the minimum size of a k-connected graph with given order and stability number. Given three positive integers n, α, k such that $n \geq \alpha + k$ (or $n \geq 1$ if $\alpha = k = 1$), let $f(n, \alpha, k)$ denote the minimum size of a k-connected graph with order n and stability number α. We say that a k-connected graph G with $|G| = n$ and $\alpha(G) = \alpha$ is (n, α, k)-*extremal* if it reaches the minimum achievable size, that is, if $||G|| = f(n, \alpha, k)$. When we consider a fixed graph G, we often simply say that G is k-extremal if it is $(|G|, \alpha(G), k)$-extremal.

Determining $f(n, \alpha, 1)$ was in fact an old problem of Ore [10] which has been settled recently independently by Christophe et al. [3] and by Gitler and Valencia [6]. They proved the following result, where $t(n, \alpha)$ is the size of the Turán graph $T(n, \alpha)$.

Proposition 1 [3, 6]. Let n, α be two positive integers such that $n \geq \alpha + 1$ (or $n \geq 1$ if $\alpha = 1$). Then $f(n, \alpha, 1) = t(n, \alpha) + \alpha - 1$.

The plan of our paper is as follows. Section 2 is devoted to definitions and background results. In Section 3, we give a short proof of the above cited result together with a characterization of $(n, \alpha, 1)$-extremal graphs. In Section 4, we determine the value of $f(n, \alpha, 2)$ and characterize $(n, \alpha, 2)$-extremal graphs. In Section 5 we use the preceding results to generalize a theorem of Brouwer related to Turán's Theorem, and we end with some remarks about $f(n, \alpha, k)$ for $k \geq 3$ in Section 6.

2. PRELIMINARIES

We gather here the definitions and basic results that we need. Undefined terms and notations can be found in Diestel [4]. Note that, unlike in this reference, we consider K_1 both as a connected and 1-connected graph. We first recall a well-known result of Brooks [1].

Proposition 2 [1]. Let G be a connected graph which is not a complete graph nor an odd cycle. Then $\chi(G) \leq \Delta(G)$.

This implies

$$\Delta(G) \geq \left\lceil \frac{|G|}{\alpha(G)} \right\rceil$$

for such a graph G, since $|G| \leq \chi(G) \cdot \alpha(G)$.

An edge e of a graph G is α-*critical* if $\alpha(G - e) > \alpha(G)$. A graph is said to be α-critical if it has no isolated vertex and all its edges are α-critical. A simple property of α-critical

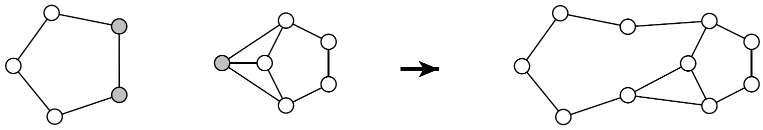

FIGURE 1. A pasting of two 2-connected graphs.

graphs is that there is always a maximum stable set avoiding a specified vertex. Indeed, let G be such a graph and $v \in V(G)$ one of its vertices. If w is any neighbor of v, then, by definition, there is a stable set S of size $\alpha(G)+1$ in $G - vw$. Now the set $S \setminus \{v\}$ is clearly a maximum stable set of G. We note also that any connected α-critical graph is 2-connected, except for K_2 (see Lovász and Plummer [9]).

One of the first results concerning α-critical graphs is the following bound given by Erdős and Gallai [5].

Proposition 3 [5]. Let G be a connected α-critical graph. Then $|G| - 2\alpha(G) \geq 0$, with equality if and only if G is isomorphic to K_2.

We now describe a construction involving two 2-connected graphs G_1 and G_2. Choose an edge $uv \in E(G_1)$ and a vertex $w \in V(G_2)$. Take the disjoint union of G_1 and G_2, and link every neighbor of w to exactly one vertex of $\{u, v\}$, ensuring that u and v are each chosen at least once, and then remove the vertex w and the edge uv. The resulting graph G is said to be obtained by *pasting* G_2 onto G_1. See Figure 1 for an illustration.

This construction has the following easy properties: The graph G is 2- but not 3-connected and $\alpha(G) \leq \alpha(G_1) + \alpha(G_2)$, with equality if G_1 or G_2 is α-critical.

Wessel [12] proved the following result (see also Lovász [8]).

Proposition 4 [12]. Let G_1, G_2 be two 2-connected α-critical graphs. Then any pasting of G_2 onto G_1 is again α-critical. Moreover, any 2-connected α-critical graph which is not 3-connected can be obtained from this construction.

For an α-critical graph G with connectivity 2 and $|G| \geq 4$, we say that (G_1, G_2) is a *Wessel pair* of G if G can be obtained by pasting G_2 onto G_1. A Wessel pair (G_1, G_2) is said to be *leftmost* (resp. *rightmost*) if G_1 (resp. G_2) has no cutset of size 2. Notice that, among all Wessel pairs of G, there is always one which is leftmost or rightmost.

A particular case of Wessel's construction arises when the second graph is a triangle: Then an edge of G_1 is replaced with a path of length 3. We say that any graph obtained using this operation finitely many times is an *odd subdivision* of the original graph. An odd subdivision is said to be *proper* when it is not isomorphic to the original graph.

Let G be a graph and $G_1, G_2, \ldots, G_k$ its components. A *tree-linking* (*cycle-linking*) of G is any graph that can be obtained by adding $k-1$ edges (resp. k edges) to G in such a way that the resulting graph is connected (resp. 2-connected) and has the same stability number as G, i.e., $\sum_{1 \leq i \leq k} \alpha(G_i)$. Note that the last condition is not restrictive when G is α-critical. This is easily seen using the fact that, in an α-critical graph, there is always a maximum stable set avoiding a given vertex. Note also that $T(n, \alpha)$ is α-critical when $n \geq 2\alpha$.

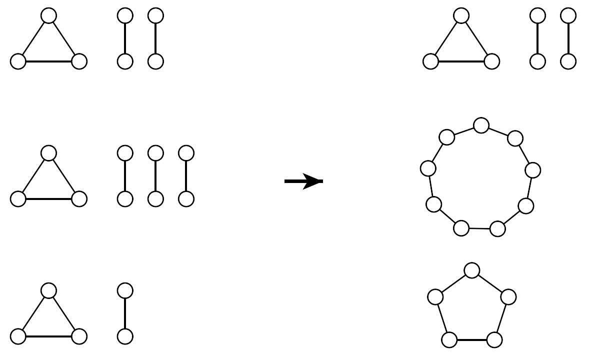

FIGURE 2. From $T(21,9)$ (left) to a possible twisted $T(21,9)$ (right).

A graph G with order n and stability number α is said to be a *twisted* $T(n,\alpha)$ if either G is isomorphic to $T(n,\alpha)$, or $2\alpha < n < 3\alpha$ and G can be obtained from $T(n,\alpha)$ using the following construction: For every $1 \le i \le k$, replace j_i copies of K_2 and one of K_3, by a copy of C_{3+2j_i}, where k and the j_i's are positive integers such that $k \le \min\{k_2, k_3\}$ and $j_1 + \cdots + j_k \le k_2$ (k_2, k_3 denote the number of copies of K_2 and K_3 respectively). See Figure 2 for an illustration.

3. THE CONNECTED CASE

We prove in this section an extended version of Proposition 1 which provides also a characterization of $(n,\alpha,1)$-extremal graphs.

Proposition 5. If n, α are two positive integers such that $n \ge \alpha + 1$ (or $n \ge 1$ if $\alpha = 1$), then

(a) $f(n,\alpha,1) = t(n,\alpha) + \alpha - 1$;

(b) a graph G is $(n,\alpha,1)$-extremal if and only if G is a tree-linking of a twisted $T(n,\alpha)$.

Note that the case $n \le 2\alpha$ is obvious: In this case, an $(n,\alpha,1)$-extremal graph is simply a tree on n vertices with stability number α.

Proof. Any tree-linking of a twisted $T(n,\alpha)$ has $t(n,\alpha)+\alpha-1$ edges and, by definition, is connected with stability number α. Thus it suffices to show that every $(n,\alpha,1)$-extremal graph G is a tree-linking of a twisted $T(n,\alpha)$. We prove it for every α, by induction on n. The case $n = 1$ clearly holds, so we assume $n \ge 2$ and that the claim is true for a strictly smaller number of vertices.

Every edge of G which is not α-critical must clearly be a bridge. If G has a non α-critical edge $e \in E(G)$, then removing e yields a graph having two components G_1,

G_2. Let $n_1 = |G_1|, n_2 = |G_2|$ and $\alpha_1 = \alpha(G_1), \alpha_2 = \alpha(G_2)$. Then $\alpha_1 + \alpha_2 = \alpha$ and the induction hypothesis yields

$$\begin{aligned} t(n,\alpha) + \alpha - 1 &\geq ||G|| \\ &= 1 + ||G_1|| + ||G_2|| \\ &\geq 1 + t(n_1,\alpha_1) + \alpha_1 - 1 + t(n_2,\alpha_2) + \alpha_2 - 1 \\ &\geq t(n,\alpha) + \alpha - 1, \end{aligned}$$

and so equality holds. In particular, G_1 and G_2 are 1-extremal and $t(n,\alpha) = t(n_1,\alpha_1) + t(n_2,\alpha_2)$. The latter implies that $T(n_1,\alpha_1) \cup T(n_2,\alpha_2)$ is isomorphic to $T(n,\alpha)$. Now, as G_1, G_2 are tree-linkings of a twisted $T(n_1,\alpha_1)$ and a twisted $T(n_2,\alpha_2)$ respectively, it follows that G is a tree-linking of a twisted $T(n,\alpha)$.

Thus we may assume that G is α-critical. We now show that G must be a complete graph or an odd cycle, which is a twisted $T(n,\alpha)$ and so completes the proof. Arguing by contradiction, assume that this does not hold and let $v \in V(G)$ be a vertex with maximum degree. Brooks' Theorem yields $\deg(v) \geq \lceil n/\alpha \rceil$. Since G is a connected α-critical graph, the graph $G-v$ is connected and $\alpha(G-v) = \alpha$, so the induction hypothesis gives

$$||G|| \geq \lceil n/\alpha \rceil + ||G-v|| \geq \lceil n/\alpha \rceil + t(n-1,\alpha) + \alpha - 1 = t(n,\alpha) + \alpha,$$

which implies that G is not 1-extremal, a contradiction (in the last equality we used the fact that $t(n-1,\alpha) + \lceil n/\alpha \rceil = t(n,\alpha) + 1$ for $n \geq 2$, which is easily deduced from the structure of Turán graphs). ∎

4. THE 2-CONNECTED CASE

Note first that, as for the $(n,\alpha,1)$-extremal ones, the structure of $(n,\alpha,2)$-extremal graphs with $n \leq 2\alpha$ is straightforward. Indeed, let G be such a graph. Then G has a stable set $S \subseteq V(G)$ with $|S| = \alpha$. Every vertex $v \in S$ has degree at least 2, since G is 2-connected, thus $||G|| \geq 2\alpha$. Now it is easily seen that there always exists a 2-connected graph on n vertices with stability number α and 2α edges. In particular, such a graph must be bipartite. Consequently, when $n \leq 2\alpha$ a 2-connected graph G with order n and stability number α is 2-extremal if and only if it is bipartite and $||G|| = 2\alpha$.

We now consider the case $n \geq 2\alpha + 1$. Define $\mathscr{H}$ as the unique graph that can be obtained by pasting a copy of K_4 onto another one.

Proposition 6. If n, α are two positive integers such that $n \geq 2\alpha + 1$, then

(a) $f(n,\alpha,2) = \begin{cases} t(n,\alpha) + \alpha - 1 & \text{if } \alpha = 1 \text{ or } n - 2\alpha = 1, \\ t(n,\alpha) + \alpha & \text{otherwise;} \end{cases}$

(b) a graph G is $(n,\alpha,2)$-extremal if and only if G satisfies one of the following conditions:

(i) G is a cycle-linking of a twisted $T(n, \alpha)$;
(ii) G is an odd subdivision of K_4;
(iii) G is isomorphic to $\mathscr{H}$.

Proof. First, note that the value for $f(n, \alpha, 2)$ follows from Proposition 5. Indeed, we have $f(n, \alpha, 2) \leq t(n, \alpha) + \alpha$, since a cycle-linking of $T(n, \alpha)$ has the latter size. Moreover, by Proposition 5, the only $(n, \alpha, 1)$-extremal graphs which are 2-connected are the complete graphs and the odd cycles (for which we have $\alpha = 1$ and $n - 2\alpha = 1$, respectively). It remains to show part (b) of the claim.

If G is as in (i), (ii) or (iii), then G is clearly 2-connected and it is easily checked that $||G|| = f(n, \alpha, 2)$.

Assume now that G is $(n, \alpha, 2)$-extremal. We prove the claim for every α, by induction on n. For $n = 3$ it is clearly true, so we assume $n \geq 4$ and that the claim holds for any strictly smaller number of vertices.

By Proposition 5, the cases $\alpha = 1$ and $n - 2\alpha = 1$ are done, so we may assume $\alpha \geq 2$ and $n - 2\alpha \geq 2$.

If G has a non α-critical edge $e \in E(G)$, then $G - e$ has connectivity 1 (otherwise, G would not be 2-extremal). Thus $G - e$ is an $(n, \alpha, 1)$-extremal graph with connectivity 1, that is, a tree-linking of a twisted $T(n, \alpha)$ by Proposition 5. Now, since G is 2-connected, it follows that G is a cycle-linking of the same twisted $T(n, \alpha)$.

We may thus suppose that G is α-critical. We first show that G has connectivity 2. Since $\alpha \geq 2$ and $n - 2\alpha \geq 2$, there exists a vertex $v \in V(G)$ with degree $\deg(v) \geq \lceil n/\alpha \rceil$ (by Brooks' Theorem). Assume that $G - v$ is 2-connected. If $n \geq 2\alpha + 3$, since $\alpha(G - v) = \alpha$, we get by induction

$$||G|| \geq \lceil n/\alpha \rceil + ||G - v|| \geq \lceil n/\alpha \rceil + t(n-1, \alpha) + \alpha = t(n, \alpha) + \alpha + 1,$$

thus G is not $(n, \alpha, 2)$-extremal, a contradiction. If $n = 2\alpha + 2$, the same argument proves that $G - v$ is an odd cycle on at least 5 vertices and that $\deg(v) = 3$. It follows that G has connectivity 2 in both cases.

Using Proposition 4, let (G_1, G_2) be a Wessel pair of G. The graph G_1 must be 2-extremal, otherwise pasting G_2 on an $(|G_1|, \alpha(G_1), 2)$-extremal graph would yield a graph G' with $\alpha(G') = \alpha$ but $||G'|| < ||G||$. Similarly, G_2 must also be 2-extremal. By Proposition 3, we have $|G_1| - 2\alpha(G_1) \geq 1$, $|G_2| - 2\alpha(G_2) \geq 1$, so the induction hypothesis applies on G_1 and G_2.

There are two possible cases, according as (G_1, G_2) is rightmost or leftmost. In the first case, the induction hypothesis implies that G_2 is a complete graph on $l \geq 3$ vertices. Moreover, G_1 cannot be isomorphic to $\mathscr{H}$. Indeed, a cycle-linking of the graph $T(6, 3) \cup G_2$ has at least one edge less than a pasting of G_2 onto $\mathscr{H}$. In particular, any proper odd subdivision of $\mathscr{H}$ is not 2-extremal. Thus G_1 is either an odd cycle or an odd subdivision of K_4.

If $l = 3$, then G is an odd subdivision of G_1 and thus satisfies the claim.

Suppose that $l = 4$. Then G_1 must be an odd cycle, otherwise G has the same size and stability number as an odd subdivison of $\mathscr{H}$ on the same number of vertices, which is not 2-extremal by the above remark. Therefore, G is an odd subdivision of K_4 and thus satisfies the claim.

Assume now $l \geq 5$. It is then easily checked that a cycle-linking of $G_1 \cup K_{l-1}$ has fewer edges than G, implying that the latter graph cannot be 2-extremal. This ends the case where (G_1, G_2) is rightmost.

We may thus assume that (G_1, G_2) is leftmost and that there exists no rightmost Wessel pair of G. The induction hypothesis implies that G_1 is a complete graph. The graph G_2 cannot be a complete graph nor an odd cycle, otherwise there would exist a rightmost Wessel pair of G. Thus G_2 is either a proper odd subdivision of K_4 or isomorphic to $\mathscr{H}$.

Suppose that G_1 is a triangle. Then G_2 cannot be isomorphic to $\mathscr{H}$ since G would then have the same order, size and stability number as a proper odd subdivision of $\mathscr{H}$. Thus G_2 is an odd subdivision of K_4 and then G itself is an odd subdivision of K_4 and so the claim is verified.

Consider now the case $|G_1| \geq 4$. Then G has the same order, size and stability number as a pasting of G_1 onto G_2, which by the above arguments is not 2-extremal. ∎

5. A GENERALIZATION OF A THEOREM OF BROUWER

Brouwer [2] proved a generalization of Turán's Theorem. He showed that, in a graph G with stability number α and order $n > 2\alpha$, the vertex set $V(G)$ can be partitioned into α cliques whenever

$$||G|| \leq t(n, \alpha) + \left\lfloor \frac{n}{\alpha} \right\rfloor - 2. \tag{1}$$

Notice that this implies Propositions 5 and 6 for $n \geq \alpha(\alpha+1)$ and $n \geq \alpha(\alpha+2)$, respectively. We also note that Brouwer's Theorem can be deduced from a more recent result due to Hanson and Toft [7].

In this section we improve (1) for connected and 2-connected graphs. The proofs follow Brouwer's original approach, combined with some ideas from the preceding sections. For convenience, the number of edges in the graph induced by a clique C will be denoted by the shorthand $||C||$.

Lemma 1. Let G be a graph of order n and suppose that $C_1, \ldots, C_\beta$ is a partition of $V(G)$ into β cliques, with $\beta \geq 2$. Then $\left(\sum_{j=1}^{\beta} ||C_j||\right) + |C_i| \geq t(n, \beta) + \lfloor n/\beta \rfloor$ for every $1 \leq i \leq \beta$.

Proof. Let $m = \lfloor n/\beta \rfloor$ and $i \in \{1, \ldots, \beta\}$. If $|C_i| \geq m$, the claim clearly holds. If $|C_i| = m - s$ $(s > 0)$, then

$$\sum_{j=1}^{\beta} ||C_j|| \geq t(n, \beta) - \binom{s}{2} - s(m-s) + sm = t(n, \beta) + \frac{s^2 + s}{2}.$$

Hence

$$\left(\sum_{j=1}^{\beta} ||C_j||\right) + |C_i| \geq t(n,\beta) + \frac{s^2+s}{2} + m - s = t(n,\beta) + \frac{s^2-s}{2} + m,$$

which is minimal for $s = 1$, and so the claim follows. ∎

Lemma 2. Let G be a connected graph with order n such that $||G|| \leq t(n,\beta) + \lfloor n/\beta \rfloor + \beta - 3$. Then there is at most one partition of $V(G)$ into β cliques.

Note that the bound on $||G||$ is best possible for $\beta \geq 2$. Indeed, consider the graph $T(n,\beta)$. If we choose a smallest component G', link all its $\lfloor n/\beta \rfloor$ vertices to the same arbitrary vertex outside G' and add $\beta - 2$ edges to make the resulting graph G connected, then G has exactly $t(n,\beta) + \lfloor n/\beta \rfloor + \beta - 2$ edges but has two distinct partitions into β cliques.

Proof. Assume that there are two distinct partitions $C_1, \ldots, C_\beta$ and $C'_1, \ldots, C'_\beta$ of $V(G)$ into β cliques. We claim that $||G|| \geq t(n,\beta) + \lfloor n/\beta \rfloor + \beta - 2$.

Set $m = \lfloor n/\beta \rfloor$. First, suppose that there exist $i, j \in \{1, \ldots, \beta\}$ such that $C_i \subsetneq C'_j$ or $C'_i \subsetneq C_j$. W.l.o.g. we may assume $C_i \subsetneq C'_j$. Let $I = \{k \in \{1, \ldots, \beta\} : |C'_j \cap C_k| > 0\}$ and $\gamma = |I|$.

Every vertex of $C'_j \setminus C_i$ must be adjacent to all vertices of C_i. Thus, when $C'_j \cap C_k \neq \emptyset$ there is a vertex of C_k which is adjacent to all the vertices of C_i. Since G is connected, we get

$$||G|| \geq \left(\sum_{k=1}^{\beta} ||C_k||\right) + (\gamma - 1)|C_i| + \beta - \gamma = \left(\sum_{k=1}^{\beta} ||C_k||\right) + \gamma(|C_i| - 1) + \beta - |C_i|.$$

The last expression is minimum for the smallest possible value of γ, that is, $\gamma = 2$. Therefore $||G|| \geq \left(\sum_{k=1}^{\beta} ||C_k||\right) + |C_i| + \beta - 2$. Using Lemma 1, we obtain $||G|| \geq t(n,\beta) + m + \beta - 2$, and so the claim holds.

For some $i \in \{1, \ldots, \beta\}$, we have $C_i \neq C'_j$ for all $1 \leq j \leq \beta$. By the above argument, we may suppose $|C_i \setminus C'_j| \geq 1$ and $|C'_j \setminus C_i| \geq 1$ for all $1 \leq j \leq \beta$. Let $J = \{j \in \{1, \ldots, \beta\} : |C'_j \cap C_i| > 0\}$. There are two possible cases, according as there exists $j \in J$ such that $C'_j \setminus C_i = \{v\}$ for some $v \in V(G)$ or not.

In the first case we construct a graph G' of order n as follows: Starting with G, we add for every vertex $w \in C_i \setminus C'_j$ the edge vw and delete the edges wx where $x \in V(G) \setminus (C_i \cup C'_j)$. The sequence $C_1, \ldots, C_\beta$ is a partition of $V(G')$ into β cliques, and there is another one, say $C^*_1, \ldots, C^*_\beta$, with the property that there exists a clique $C^*_j \supsetneq C_i$. Now we get by the beginning of the proof $||G|| \geq ||G'|| \geq t(n,\beta) + m + \beta - 2$.

In the second case we have $|C'_j \setminus C_i| \geq 2$ for every $j \in J$. Consider a clique C'_j with $j \in J$. Let $v \in C'_j \setminus C_i$. Remark that v is adjacent to every vertex of $C_i \cap C'_j$. Let $L_j = \{k \in \{1, \ldots, \beta\} : |C'_j \cap C_k| > 0\}$. Since $|C'_j \setminus C_i| \geq 2$, there are at least $|L_j| - 1$ edges (in fact even $\binom{|L_j|-1}{2}$) which are not of the form vw with $w \in C_i$, and which are not included in a clique C_k. Let $L = \cup_{j\in J} L_j$. We have $|L| - 1 \leq \sum_{j\in J}(|L_j| - 1)$, because $i \in L_j$ for every $j \in J$. So, as G is connected, the number of edges of G which are not included in any C_l for $1 \leq l \leq \beta$ is at least $|C_i| + \sum_{j\in J}(|L_j| - 1) + \beta - |L| \geq |C_i| + \beta - 1$. This implies $||G|| \geq \left(\sum_{k=1}^{\beta} ||C_k||\right) + |C_i| + \beta - 1$. By Lemma 1, it follows $||G|| \geq t(n, \beta) + m + \beta - 1$. ∎

Lemma 3. Let G be a connected graph with order $n \geq 2\beta$, $\Delta(G) \geq \lceil n/\beta \rceil$ and $||G|| \leq t(n, \beta) + \lfloor n/\beta \rfloor + \beta - 3$ for some positive integer $\beta \geq 2$. Suppose that for every vertex $v \in V(G)$ of degree at least $\lfloor n/\beta \rfloor$ the graph $G - v$ is connected and $V(G - v)$ can be partitioned into β cliques. Then $V(G)$ can also be partitioned into β cliques.

Proof. Arguing by contradiction, we suppose that $V(G)$ cannot be partitioned into β cliques. Let $v, w \in V(G)$ be two vertices of degree at least $\lceil n/\beta \rceil$ and at least $m = \lfloor n/\beta \rfloor$ respectively. By hypothesis, $V(G - v), V(G - w)$ can be partitioned into β cliques $C_1^v, \ldots, C_\beta^v$ and $C_1^w, \ldots, C_\beta^w$ respectively.

Let i_0 and i_1 be the indices such that $w \in C_{i_0}^v$ and $v \in C_{i_1}^w$. W.l.o.g. we have $|C_{i_0}^v| \geq 2$. Indeed, if $C_{i_0}^v = \{w\}$, then we consider an arbitrary neighbor w' of w in $G - v$ and we let j be the index such that $w' \in C_j^v$. If $|C_j^v| = 1$, then by merging $C_{i_0}^v$ and C_j^v one easily finds a partition of $V(G)$ into β cliques. Otherwise, moving w' from C_j^v to $C_{i_0}^v$ yields a partition of $V(G - v)$ into β cliques with the desired property. Similarly, we may assume $|C_{i_1}^w| \geq 2$. Thus $C_1^v \setminus \{w\}, \ldots, C_\beta^v \setminus \{w\}$ and $C_1^w \setminus \{v\}, \ldots, C_\beta^w \setminus \{v\}$ are both partitions of $V(G - \{v, w\})$ into β cliques. Moreover,

$$\begin{aligned} ||G - \{v, w\}|| &\leq ||G|| - \left\lceil \tfrac{n}{\beta} \right\rceil - \left\lfloor \tfrac{n}{\beta} \right\rfloor + 1 \\ &\leq t(n, \beta) + \left\lfloor \tfrac{n}{\beta} \right\rfloor + \beta - 3 - \left\lceil \tfrac{n}{\beta} \right\rceil - \left\lfloor \tfrac{n}{\beta} \right\rfloor + 1 \\ &= t(n - 2, \beta) + \left\lfloor \tfrac{n-2}{\beta} \right\rfloor + \beta - 3. \end{aligned}$$

By Lemma 2, we may assume w.l.o.g. $C_i^v \setminus \{w\} = C_i^w \setminus \{v\}$ for $1 \leq i \leq \beta$. We have $i_0 = i_1$ and $vw \notin E(G)$, because otherwise there would be a partition of $V(G)$ into β cliques. Therefore, no vertex of degree at least m is adjacent to v. In particular, every vertex of $C_{i_0}^v \setminus \{w\}$ has degree less than m. It follows that there are no clique C_i^v which contains $m + 1$ vertices.

Now, there are only two possibilities: Either $n = m\beta + 1$ and $|C_i^v| = m$ for $1 \leq i \leq \beta$, or $n = m\beta$, $|C_i^v| = m$ for $1 \leq i \leq \beta$, $i \neq i_0$ and $|C_{i_0}^v| = m - 1$. In the first case, no vertex is adjacent to v, a contradiction. In the second case, the only vertices of G that can be adjacent to v are those from $C_{i_0}^v$, but then the degree of v cannot be at least $\lceil n/\beta \rceil = m$, a contradiction. ∎

Proposition 7. Let G be a connected graph with order n and stability number α such that $n \geq 3\alpha$ and $||G|| \leq t(n, \alpha) + \lfloor n/\alpha \rfloor + \alpha - 4$. Then $V(G)$ can be partitioned into α cliques.

Note that the bound on $||G||$ is best possible when $\alpha \geq 2$. Indeed, consider the graph $T(n, \alpha)$ and denote by G_1, G_2 respectively one of its smallest component and another component (thus $|G_1| = \lfloor n/\alpha \rfloor$). Add a new vertex v, link it to all vertices of G_1, denote by G_1' this modified component, and pick an edge e of G_2. Replace the two components G_1', G_2 by a pasting of the latter onto the former using e and v, and denote by $\widetilde{T}(n, \alpha)$ the resulting graph. Any tree-linking of the latter graph has $t(n, \alpha) + \lfloor n/\alpha \rfloor + \alpha - 3$ edges but cannot be partitioned into α cliques.

Proof. We proceed by induction on n, and for a fixed n, we proceed by induction on $||G||$. If $3\alpha \leq n < 4\alpha$, the claim follows from Proposition 5, so we assume $n \geq 4\alpha$ and that the claim is true for every α, for a strictly smaller number of vertices. If $||G|| = f(n, \alpha, 1)$, i.e., if G is $(n, \alpha, 1)$-extremal, the result follows again by Proposition 5. We thus suppose $||G|| \geq f(n, \alpha, 1) + 1$.

If G has an edge which is not α-critical nor a bridge, then we find the claim by a direct induction on $||G||$. Therefore, we may assume that every edge of G which is not α-critical is a bridge. If G has a non α-critical edge $e \in E(G)$, then by removing e, we get a graph having two components G_1, G_2. Let $n_1 = |G_1|, n_2 = |G_2|$ and $\alpha_1 = \alpha(G_1), \alpha_2 = \alpha(G_2)$. Then $\alpha_1 + \alpha_2 = \alpha$. We now prove $n_1 \geq 3\alpha_1$, $n_2 \geq 3\alpha_2$, $||G_1|| \leq t(n_1, \alpha_1) + \lfloor n_1/\alpha_1 \rfloor + \alpha_1 - 4$ and $||G_2|| \leq t(n_2, \alpha_2) + \lfloor n_2/\alpha_2 \rfloor + \alpha_2 - 4$. The claim will then follow by induction on n.

By contradiction, suppose that $||G_1||$ or $||G_2||$ does not verify the above bounds. W.l.o.g. we may assume that this happens for $||G_1||$. The graph $T(n_1, \alpha_1) \cup T(n_2, \alpha_2)$ has clearly a partition into α cliques, one of them having exactly $\lfloor n_1/\alpha_1 \rfloor$ vertices. By Lemma 1, $t(n_1, \alpha_1) + t(n_2, \alpha_2) + \lfloor n_1/\alpha_1 \rfloor \geq t(n, \alpha) + \lfloor n/\alpha \rfloor$. Then by Proposition 5, it follows

$$\begin{aligned} ||G|| &= ||G_1|| + ||G_2|| + 1 \\ &\geq t(n_1, \alpha_1) + \left\lfloor \tfrac{n_1}{\alpha_1} \right\rfloor + \alpha_1 - 3 + t(n_2, \alpha_2) + \alpha_2 - 1 + 1 \\ &\geq t(n, \alpha) + \left\lfloor \tfrac{n}{\alpha} \right\rfloor + \alpha - 3, \end{aligned}$$

a contradiction. Thus the claimed bounds on $||G_1||, ||G_2||$ hold. Now, as G_1, G_2 are both connected, $n_1 < 3\alpha_1$ or $n_2 < 3\alpha_2$ would contradict the lower bound on the number of edges given in Proposition 5.

Thus we may assume that G is α-critical, and so 2-connected. In particular, for every vertex $v \in V(G)$ of degree at least $\lfloor n/\alpha \rfloor$, the graph $G - v$ is connected and $||G - v|| \leq t(n, \alpha) + \alpha - 4 = t(n-1, \alpha) + \lfloor (n-1)/\alpha \rfloor + \alpha - 4$. Using the induction hypothesis on n, we find that $V(G - v)$ can be partitioned into α cliques. Clearly, the claim holds when $\alpha = 1$. Assume now $\alpha \geq 2$. Since $n \geq 4\alpha$, the graph G is not a complete graph nor an odd cycle and thus, by Brooks' Theorem, it has a vertex of degree at least $\lceil n/\alpha \rceil$. So by Lemma 3, $V(G)$ can also be partitioned into α cliques. ∎

We note that, contrary to what may seem in view of the above proof, it is never possible to partition the vertex set of a connected α-critical with stability number $\alpha \geq 2$ into α cliques.

Proposition 8. Let G be a 2-connected graph with order n and stability number α such that $n \geq 3\alpha \geq 9$ and $||G|| \leq t(n, \alpha) + \lfloor n/\alpha \rfloor + \alpha - 3$. Then $V(G)$ can be partitioned into α cliques.

Note that the bound on $||G||$ is again best possible: Any cycle-linking of $\widetilde{T}(n,\alpha)$ has $t(n,\alpha)+\lfloor n/\alpha\rfloor+\alpha-2$ edges but cannot be partitioned into α cliques.

Proof. We proceed by induction on n. If $3\alpha \le n < 4\alpha$, the claim follows from Proposition 6, so we assume $n \ge 4\alpha$ and that the claim is true for $n-1$.

If $||G|| \le t(n,\alpha)+\lfloor n/\alpha\rfloor+\alpha-4$ or if G has a non α-critical edge, then the result follows from Proposition 7. So we may assume $||G|| = t(n,\alpha)+\lfloor n/\alpha\rfloor+\alpha-3$ and that G is α-critical.

First, suppose that G is 3-connected. Then for every vertex $v \in V(G)$ of degree at least $\lfloor n/\alpha\rfloor$, the graph $G-v$ is 2-connected and $||G-v|| \le t(n,\alpha)+\alpha-3 = t(n-1,\alpha)+\lfloor (n-1)/\alpha\rfloor+\alpha-3$. By the induction hypothesis on n, we have that $V(G-v)$ can be partitioned into α cliques. The graph G is not a complete graph nor an odd cycle, as $\alpha \ge 3$ and $n \ge 4\alpha$. By Brooks' Theorem, there exists a vertex of degree at least $\lceil n/\alpha\rceil$ and, by Lemma 3, $V(G)$ can also be partitioned into α cliques.

Now, assume that G has connectivity 2 and let (G_1,G_2) be a Wessel pair of G. Let $v,v' \in V(G_1)$, $w,w' \in V(G_2)$ be two distinct vertices of respectively G_1 and G_2, and let G' be the graph of order $n+1$, consisting of the disjoint union of G_1 and G_2. We have $\alpha(G'+vw)=\alpha(G'+v'w')=\alpha$, $n+1 \ge 4\alpha+1$ and

$$\begin{aligned} ||G'+vw|| = ||G'+v'w'|| &= ||G_1||+||G_2||+1 = ||G||+2 \\ &\le t(n,\alpha)+\left\lfloor \tfrac{n}{\alpha}\right\rfloor+\alpha-1 \\ &= t(n+1,\alpha)+\alpha-1 \\ &\le t(n+1,\alpha)+\left\lfloor \tfrac{n+1}{\alpha}\right\rfloor+\alpha-4 \end{aligned}$$

By Proposition 7, $V(G'+vw)$ and $V(G'+v'w')$ can be both partitioned into α cliques. Lemma 2 gives also that such a partition is unique for $V(G'+\{vw,v'w'\})$, so we deduce that $V(G_1)$ and $V(G_2)$ can be partitioned into respectively $\alpha(G_1)$ and $\alpha(G_2)$ cliques. But this is possible only if both graphs are complete, implying $\alpha=2$, a contradiction. ∎

6. REMARKS ABOUT THE *K*-CONNECTED CASE

It is not difficult to determine $f(n,\alpha,k)$ for $k \ge 3$ when $n \le 2\alpha$: By an argument similar to the case $k=2$, we have that a k-connected graph G with order n and stability number α is k-extremal if and only if it is bipartite and $||G|| = k\alpha$.

The case $n \ge 2\alpha$ is not so clear. If $2\alpha \le n \le k\alpha$, then $f(n,\alpha,k) \ge \lceil nk/2\rceil$, as every vertex of a k-connected graph has degree at least k. If $n \ge k\alpha$, we have $f(n,\alpha,k) \le t(n,\alpha)+\lceil k\alpha/2\rceil$, since the graph $T(n,\alpha)$ can be made k-connected by the adjunction of $\lceil k\alpha/2\rceil$ edges without decreasing its stability number. We are tempted to believe that these two bounds are actually the exact value of $f(n,\alpha,k)$.

Conjecture 1. Let n,α,k be three positive integers such that $n \ge 2\alpha$, $n \ge \alpha+k$, $\alpha \ge 2$ and $k \ge 3$. Then

$$f(n,\alpha,k) = \begin{cases} \lceil nk/2\rceil & \text{if } n \le k\alpha, \\ t(n,\alpha)+\lceil k\alpha/2\rceil & \text{otherwise.} \end{cases}$$

■

Notice that this conjecture generalizes the 2-connected case when $n \geq 2\alpha + 2$. Moreover, for $n = k\alpha$, we have $f(n, \alpha, k) = t(n, \alpha) + \lceil k\alpha/2 \rceil = \lceil nk/2 \rceil$.

We note also that, using Proposition 8, this conjecture is true when $\alpha \geq 3$ and $n \geq \lceil (k-2)\alpha/2 \rceil \alpha + 2\alpha$. Indeed, assume that G is an (n, α, k)-extremal graph with size at most $t(n, \alpha) + \lceil k\alpha/2 \rceil - 1$. We have $||G|| \leq t(n, \alpha) + \lceil k\alpha/2 \rceil - 1 \leq t(n, \alpha) + \lfloor n/\alpha \rfloor + \alpha - 3$, and so $V(G)$ can be partitioned into α cliques $C_1, \dots, C_\alpha$. Since G is k-connected, $||G|| \geq (\sum_{i=1}^{\alpha} ||C_i||) + \lceil k\alpha/2 \rceil \geq t(n, \alpha) + \lceil k\alpha/2 \rceil$, a contradiction.

Moreover, Conjecture 1 is true for $\alpha = 2$ and $n \geq 2k$. Using Proposition 7, we find as above that the conjecture is verified when $n \geq 2k + 2$. We also know that it is true if $n = 2k$, so it remains to consider the case $n = 2k+1$. Suppose that G is an $(2k+1, 2, k)$-extremal graph. Then by Brooks' Theorem it has a vertex $v \in V(G)$ of degree at least $\lceil (2k+1)/2 \rceil = k+1$. If G is α-critical, then $G-v$ is $(k-1)$-connected, $n-1 = 2(k-1)+2$ and $\alpha(G-v) = \alpha$, hence $||G|| \geq k+1+||G-v|| \geq k+1+t(2k, 2)+k-1 = t(2k+1, 2)+k$. If G is not α-critical, then it has a non α-critical edge $e \in E(G)$. Since G is k-connected, $G - e$ is $(k-1)$-connected, and so $||G|| = 1 + ||G - e|| \geq 1 + t(2k+1, 2) + k - 1 = t(2k+1, 2) + k$.

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References

[1] R. L. Brooks, *On colouring the nodes of a network*, Proc. Cambridge Philos. Soc. **37** (1941), 194–197.

[2] A. E. Brouwer, *Some lotto numbers from an extension of Turán's theorem*, Afdeling Zuivere Wiskunde [Department of Pure Mathematics], vol. 152, Mathematisch Centrum, Amsterdam, 1981.

[3] J. Christophe, S. Dewez, J.-P. Doignon, S. Elloumi, G. Fasbender, P. Grégoire, D. Huygens, M. Labbé, H. Mélot, and H. Yaman, *Linear inequalities among graph invariants: using graphedron to uncover optimal relationships*, submitted (e-print available on Optimization Online).

[4] R. Diestel, *Graph theory*, third ed., Graduate Texts in Mathematics, vol. 173, Springer-Verlag, Berlin, 2005.

[5] P. Erdős and T. Gallai, *On the minimal number of vertices representing the edges of a graph.*, Magyar Tud. Akad. Mat. Kutató Int. Közl. **6** (1961), 181–203.

[6] I. Gitler and C. E. Valencia, *Bounds for graph invariants*, arXiv:math.CO/0510387.

[7] D. Hanson and B. Toft, *k-saturated graphs of chromatic number at least k*, Ars Combin. **31** (1991), 159–164.

[8] L. Lovász, *Combinatorial problems and exercises*, second ed., North-Holland Publishing Co., Amsterdam, 1993.

[9] L. Lovász and M. D. Plummer, *Matching theory*, North-Holland Mathematics Studies, vol. 121, North-Holland Publishing Co., Amsterdam, 1986, Annals of Discrete Mathematics, 29.

[10] O. Ore, *Theory of graphs*, American Mathematical Society Colloquium Publications, Vol. XXXVIII, American Mathematical Society, Providence, R.I., 1962.

[11] P. Turán, *Eine Extremalaufgabe aus der Graphentheorie*, Mat. Fiz. Lapok **48** (1941), 436–452.

[12] W. Wessel, *Kanten-kritische Graphen mit der Zusammenhangszahl* 2, Manuscripta Math. **2** (1970), 309–334.